MAT4410 (2020 AUTUMN) MANDATORY ASSIGNMENT

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You need to upload your solution to the Canvas system, by October 30 (Friday). Solve at least one question in each section.

1. Measure theory

Problem 1. Put $X_0 = \{0, 1\}$, and consider the uniform probability measure $\mu(\{0\}) = \mu(\{1\}) = \frac{1}{2}$ on X_0. Take the product space (the Cantor space) $X = \prod_{n=1}^{\infty} X_0$ of countable copies of X_0, equipped with the product measure $\mu = \bigotimes \mu_0$. The goal is to show that (X, μ) is isomorphic to $([0, 1], m)$, where m is the (restriction of) Lebesgue measure.

(1) Give a candidate of the comparison map $f : X \to [0, 1]$. Hint: use the idea of binary expansion. It's enough to construct a map that is bijective outside of a countable subset $A \subset X$.
(2) Check $\mu(A) = m(f(A))$ for tubular sets $A \subset X$ (subsets of the form
$$A = Y_1 \times \cdots \times Y_M \times \prod_{n=M+1}^{\infty} X_0$$
for some $Y_n \subset X_0$ for $n = 1, \dots, M$).
(3) Check $f_*\mu = m$. Hint: do the sets $f(A)$ generate the σ-algebra of the Borel sets of $[0, 1]$?
(4) Can you modify this construction to have a bijective map?

Problem 2. Give examples showing the importance of assumptions of Fubini's theorem:

(1) Give a measure μ on the Borel σ-algebra of $[0, 1]$ and a bounded nonnegative Borel measurable function f on $[0, 1]^2$ such that
$$\int_{[0,1]} \int_{[0,1]} f(x, y) d\mu(x) dm(y) \neq \int_{[0,1]} \int_{[0,1]} f(x, y) dm(y) d\mu(x)$$
for the Lebesgue measure m.
(2) Give a measurable function f on $[0, 1]^2$ such that
$$\int_{[0,1]} \int_{[0,1]} f(x, y) dm(x) dm(y) \neq \int_{[0,1]} \int_{[0,1]} f(x, y) dm(y) dm(x).$$

Problem 3. Let X and Y be independent (real) random variables on some sample space $(\Omega, \mathbb{P})$. How do you model the joint distribution of (X, Y) by a product measure space?
Hint: X as a measurable map $\Omega \to \mathbb{R}$ defines a measure μ_X on $\mathbb{R}$ by $\mu_X(A) = \mathbb{P}[X(\omega) \in A]$ for Borel subsets $A \subset \mathbb{R}$.

Problem 7. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, and $f : X \to [0, \infty]$ be a nonnegative measurable function. Put
$$G_f = \{(x, y) \in X \times [0, \infty] \mid y \leq f(x)\}.$$
(This can be interpreted as the region between X-axis and the graph of f.) Show that the (restriction of) Lebesgue measure m on $[0, \infty]$ satisfies $(\mu \otimes m)(G_f) = \int f d\mu$.
Hint: what is $\int_{[0,\infty]} 1_{G_f}(x, y) dm(y)$?

2. L^p and Banach spaces

Problem 4. Let (Ω, μ) be a measure space and $1 < p < \infty$. The goal here is to understand that $L^p(\Omega, \mu)$ is *uniformly convex* as a normed vector space: for any $\epsilon > 0$ there is $\delta > 0$ such that, if $x, y \in L^p(\Omega, \mu)$ are unit vectors satisfying $\|x + y\|_p > 2 - \delta$, then $\|x - y\|_p < \epsilon$.

(1) A proof is given in [O. Hanner, On the uniform convexity of L^p and l^p, Ark. Mat. **3** (1956), 239–244.]. Extract the ingredients of the proof for the case $1 < p < 2$.

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(2) Show that $L^1(\Omega, \mu)$ and $L^\infty(\Omega, \mu)$ do *not* have the uniform convexity (unless they are 1-dimensional).
(3) Show that $L^1(\Omega, \mu)$ and $L^\infty(\Omega, \mu)$ are not (isometrically) isomorphic to a subspace of $L^p(\Omega', \mu')$ for any (Ω', μ').

Problem 5. Give example of functionals on $C_c(\mathbb{R})$ satisfying the following conditions:
(1) continuous for the norm $\|f\|_\infty$, but not for $\|f\|_p$ for any of $1 \leq p < \infty$
(2) continuous for the norm $\|f\|_2$, but not for $\|f\|_\infty$.

Extra: is there a bounded functional ψ on $C_b(\mathbb{R})$ satisfying $\psi(f) = \psi(\tau_t f)$ for all $t \in \mathbb{R}$, where $(\tau_t f)(x) = f(x+t)$?

Problem 6. Consider the quotient space $X = \ell^\infty(\mathbb{N})/c_0(\mathbb{N})$.
(1) show that there is a functional ψ of norm 1 on X satisfying $\psi((a_n)_{n\in\mathbb{N}}) = c$ if $a_n = c$ for all n. Hint: first look at the 1-dimensional subspace spanned by the constant sequences.
(2) show that $\psi((a_n)_{n\in\mathbb{N}}) = \lim a_n$ if $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence. Hint: can you describe the convergent sequences in $\ell^\infty(\mathbb{N})$ using the constant sequences and $C_0(\mathbb{N})$?

Problem 8. What is wrong with the following argument?: Let X be the space of functions on $[0,1]$ represented by polynomials with real coefficients. We make it a normed vector space by $\|p\|_\infty = \max_{0\leq t\leq 1} |p(t)|$. Consider the maps

$$D_n \colon X \to \mathbb{R}, \quad D_n(p) = \frac{d^n p}{dt^n}(1).$$

For any given $p \in X$, we have $D_n(p) = 0$ for sufficiently large n. In particular, $|D_n(p)|$ for $n = 0, 1, \dots$ are bounded. By Uniform Boudedness Principle, $\|D_n\|$ for $n = 0, 1, \dots$ are bounded (say, by a constant $C > 0$). Looking at the functions $p_n(t) = t^n$, we have $\|p_n\|_\infty = 1$ and $D_n(p_n) = n!$. So $n! \leq C$ holds for all n.