MAT 4450 (Intro). 2020.08.24 1.

Overview :

- Banach spaces
- advanced measure theory

<u>Banach</u> <u>spaces</u> : beyond Hilbert spaces

↳ complete normed vector spaces

Norm : "length" of vectors $\|f\|$

Motivation: regularity of functions

e.g.: $\frac{df(x)}{dx}$ bounded? continuous? integrable?

Completeness: allow "approximation" of vectors

Motivation: approx. construction of solution to diff. eq.

e.g. $\frac{df(x,t)}{dt} = Af(x,t)$ $f(x,t) \sim \sum_{k=0}^{N} \frac{(tA)^k}{k!} f(x,0)$

Operations on Banach spaces

- linear transform $T(f_1 + f_2) = Tf_1 + Tf_2$

  $T(af) = aTf$

  compatibility with norm: boundedness

  $\|Tf\| \leq c\|f\|$

- functionals (linear forms). $\phi(f)$

One goal : <u>integral</u> <u>representation</u>

e.g. $\phi(f) = \int f(x)\, g(x)\, dx$

When do we have this? regularity of g?

Continues in MAT4450 (spectral theory)

e.g. eigenval. of selfadjoint operators

$$\frac{d^2 f(x)}{dx^2} + x^2 f(x) = a f(x), \dots$$

Advanced measure theory.

- Product measures $\mu \otimes \nu$ for measures.

  integration of independent variables
  ↳ model by direct product
  ↳ model by measure on measurable spaces.

  Motivation: double integrals.

  $$\int \left(\int f(x,y)\, dx \right) dy = \int \left(\int f(x,y)\, dy \right) dx$$

  when can we make sense of this?

- Modes of convergence (covered in 4400?)

  Almost everywhere convergence

  $f_n(x) \to f(x)$ a.e.

  $\{x : f_n(x) \not\to f(x)\}$ has measure zero

  Convergence in measure $f_n \xrightarrow{\mu} f$

  $\forall \varepsilon > 0$: $\mu(\{x : |f_n(x) - f(x)| > \varepsilon\}) \to 0$ $(n \to \infty)$

- Signed measures

  ↳ allow $\mu(A) \in \mathbb{R}$ for $A \subset X$ measurable.

  Integral rep. of functionals on $L^p(X, \mu)$

  $$\phi(f) = \int f \cdot g \, d\mu \qquad g \in L^q(X, \mu)$$

Terminology & notation for measured spaces.

<u>Measurable space</u> $(X, \mathcal{M})$.

$\mathcal{M}$: σ-algebra on X i.e.

collection of subsets of X

"measurable"

allow taking :

- complements $A \in \mathcal{M} \Rightarrow X \setminus A \in \mathcal{M}$
- coutable unions $A_1, A_2, \cdots \in \mathcal{M}$

$$\Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{M}$$

- X itself is in $\mathcal{M}$ $(\Rightarrow \emptyset = X \setminus X \in \mathcal{M})$

<u>Measure</u> μ on $(X, \mathcal{M})$

map $\mu : \mathcal{M} \to [0, \infty]$. s.t.

- $\mu(\emptyset) = 0$
- $A_1, \cdots, A_n \in \mathcal{M}$ disjoint $\underset{i \neq j}{A_i \cap A_j = \emptyset}$

$$\Rightarrow \sum_{i=1}^{n} \mu(A_i) = \mu\left(\bigcup_{i=1}^{n} A_i\right).$$

- $A_1, A_2, \cdots \in \mathcal{M}$ $A_1 \subset A_2 \subset \cdots$

$$\Rightarrow \mu(A_i) \to \mu\left(\bigcup_{i=1}^{\infty} A_i\right) \quad (i \to \infty)$$

Examples

1. Counting measure.

   X : any set. $\mathcal{M} = \{\text{all subsets of } X\}$.

   $\mu(A) = |A|. \in \{0, 1, \cdots, \infty\}$

2. Lebesgue measure

   $X = \mathbb{R}$, $\mathcal{M} = \mathcal{B}_{\mathbb{R}} = \{\text{Borel subsets of } \mathbb{R}\}$

   (smallest σ-alg. containing $[a, b]$ $\forall a, b$)

   $\mu = m$ s.t. $m([a, b]) = b - a$ for $a < b$.

Measurable maps . $(X, \mathcal{M}) \to (Y, \mathcal{N})$

f: map from X to Y s.t. $\forall B \in \mathcal{N}$

$f^{-1}(B) \in \mathcal{M}$.

For $Y = \mathbb{R}$ take $\mathcal{N} = \mathcal{B}_{\mathbb{R}}$ implicitly

Integration $f : (X, \mathcal{M}) \to \mathbb{R}$.

μ : measure on $(X, \mathcal{M})$

- if f is nonneg. $(f(x) \geq 0 \quad \forall x \in X)$

  $\int f \, d\mu = \int f(x) \, d\mu(x)$ from approx of f by simple functions $\left(\sum_{i=1}^{n} a_i \mathbb{1}_{B_i} \right)$

  $a_i \geq 0, \ B_i \in \mathcal{M}$

- if $\int |f| \, d\mu \leq \infty$ (f is μ-integrable)

  then $\int f \, d\mu$ still makes sense.