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Product of measurabl spaces.

$(X_1, \mathcal{M}_1)$, $(X_2, \mathcal{M}_2)$ meas. sp.

⇝ if $A_i \in \mathcal{M}_i$ $(i=1,2)$ $A_1 \times A_2$ should be a measurable subset of $X_1 \times X_2$

⇝ should take

$\mathcal{M}_1 \otimes \mathcal{M}_2$ = σ-alg generated by the collection $\{ A_1 \otimes A_2 : A_i \in \mathcal{M}_i \}$

as σ-alg on $X_1 \times X_2$.

Variations 1. $(X_i, \mathcal{M}_i)$ $i = 1, \dots, N$

⇝ $\mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_N$: σ-alg on $X_1 \times \cdots \times X_N$ from $A_1 \times \cdots \times A_N \in X_1 \times \cdots \times X_N$ $A_i \in \mathcal{M}_i$.

2. $(X_i, \mathcal{M}_i)$ $i = 1, 2, \dots$

⇝ $\bigotimes_{i=1}^{\infty} \mathcal{M}_i$ on $\prod_{i=1}^{\infty} X_i$ from $\prod_{i=1}^{\infty} A_i$ for all $A_i \in \mathcal{M}_i$

Rem. enough to start from $X_1 \times \cdots \times X_{k-1} \times A_k \times X_{k+1} \times \cdots$ for all $k = 1, 2, \dots$ and all $A_k \in \mathcal{M}_k$

Corresp. with random variables

X, Y indep. random vars. modelled by $(\Omega, \mathcal{F}, \mu)$, $(\Omega', \mathcal{F}', \mu)$ measurable sp. (forget about measure now)

meas. funcs $f_X : \Omega \to \mathbb{R}$, $f_Y : \Omega' \to \mathbb{R}$.

⇝ polynomials of f_X & f_Y e.g. $f_X f_Y + f_Y^2$, …

give $\mathcal{F} \otimes \mathcal{F}'$-meas. funcs on $\Omega \times \Omega'$.

Want: "product measure" $\mu \otimes \mu'$ modelling joint distrib. of X & Y.

Lem $\overset{\text{Driver}}{(20.2)}$ $(X, \mathcal{M})$, $(Y, \mathcal{N})$ meas. sp.

$f : \underset{\underset{\mathcal{M}\otimes\mathcal{N}}{\uparrow}}{X \times Y} \to \underset{\underset{\mathcal{B}_{\mathbb{R}}}{\uparrow}}{\mathbb{R}}$ measurable func

Then for fixed $y \in Y$, the func $f_y(x) = f(x, y)$

on X is measurable.

Proof Idea Step 1 check this for

$f = 1_{A \times B}$ $A \in \mathcal{M}$, $B \in \mathcal{N}$. ($f_y = 0$ or $f_y = 1_A$).

Step 2 : Step 1 $\Rightarrow$ claim for $f = 1_E$, $E \in \mathcal{M}\otimes\mathcal{N}$

Step 3 : Step 2 $\Rightarrow$ claim for general case

Step 2 rough idea :

- compliment $E^c = (X \times Y) \setminus E$ gives $1_{E^c} = \boxed{1 - 1_E}$
- countable union $\bigcup_{i=1}^{\infty} E_i$ gives

$$1_{\bigcup_{i=1}^{\infty} E_i} = \boxed{\sup_{i=1,2,--} 1_{E_i}}$$

preserve measurability of functions.

we know $E = A \times B$ is OK. & can repeat these ops.

Formally : $\mathcal{L} = \{ E \subset X \times Y : (1_E)_y$ is $\mathcal{M}$-meas. $(1_E)_x$ is $\mathcal{N}$-meas. $\forall x, y. \}$

is a Dynkin sys., contains coll. $\{ \underset{\underset{\mathcal{M}}{\cap}}{A} \times \underset{\underset{\mathcal{N}}{\cap}}{B} \}$

$\Rightarrow$ $\mathcal{L}$ is the σ-alg gen'ed by this coll.

i.e. $\mathcal{M}\otimes\mathcal{N}$

Step 3. Simple functions : OK by Step 2.

Non neg. measurable f :

$f_1, f_2, \cdots$ simple (non neg) funcs, $f_1 \leq f_2 \leq \cdots$

$f(x, y) = \lim_{i \to +\infty} f_i(x, y)$ OK.

General f : $f = f^+ - f^-$, $f^{\pm} \geq 0$. OK. □

Notn. $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$ measure sp.

$f : X \times Y \to \mathbb{R}$ $\mathcal{M} \otimes \mathcal{N}$-meas. func.

$$\int d\mu(x) \int d\nu(y)\, f(x,y) = \int_X \left(\int_Y f(x,y)\, d\nu(y) \right) d\mu(x)$$

$$\int d\nu(y) \int d\mu(x)\, f(x,y) = \int_Y \left(\int_X f(x,y)\, d\mu(x) \right) d\nu(y)$$

Recall : $(X, \mathcal{M}, \mu)$ is <u>σ-finite</u> if

if $X = \bigcup_{i=1}^{\infty} A_i$ for some $A_i \in \mathcal{M}$ with

$\mu(A_i) < \infty$

e.g. $X = \mathbb{R}$, $\mu = m$.

Th'm (20.5) $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$: σ-finite meas. sp.

$f : X \times Y \to [0, \infty)$ measurable.

Then $F(x) = \int f(x,y)\, d\nu(y)$, $F'(y) = \int f(x,y)\, d\mu(x)$) are measurable as maps to $[0, \infty]$

Proof. Step 1 check this when $\mu(X), \nu(Y) < \infty$

& $f = 1_{A \times B}$ for some $A \in \mathcal{M}$, $B \in \mathcal{N}$.

Step 2 Step 1 ⇒ OK for $f = 1_E$, $E \in \mathcal{M} \otimes \mathcal{N}$

Step 3 Step 2 ⇒ O.K. for bounded f.

Step 4 remove $\mu(X), \nu(Y) < \infty$

Step 5 : remove boundedness of f.

More details : Step 2 & Step 3 have

comparison of $\lim f_n$ for $f_1 \le f_2 \le \cdots \to f$

and integrals.

⇝ use the dominated convergence th'm

(possible by $\mu(X), \nu(Y) < \infty$, f bdd ⇒ f integrable)

Step 4 : if $X = \bigcup_{i=1}^{\infty} A_i$ $A_1 \subset A_2 \subset \cdots$, $\mu(A_i) < \infty$

$$\int_X f(x,y)\,d\mu(x) = \underset{\uparrow}{\sup} \underbrace{\int_{A_i} f(x,y)\,d\mu(x)}_{\text{know measurability in } y}.$$

preserves measurability

Step 5 : $f = \sup_{N \to \infty} \underbrace{\mathbb{1}_{E_N} \cdot f}_{\text{bdd.}}$

for $E_N = \{(x,y) \in X \times Y : f(x,y) \le N\}$ ☒