Last time :

- direct product of measurable spaces

  $A \subset X$, $B \subset Y$ measurable $\Rightarrow$ $A \times B \subset X \times Y$ meas.

- $f : X \times Y \to [0, \infty]$ measurable

  μ measure on X, ν meas. on Y

  $\Rightarrow$ $F(x) = \int f(x,y)\, d\nu(y)$ meas. in x

  $F'(y) =$ ($x \leftrightarrow y$, $\mu \leftrightarrow \nu$ in above def) ...

- prod. meas. $\mu \otimes \nu\,(E) = \int d\mu(x) \int d\nu(y)\, 1_E(x,y)$

Want : $\int d\mu(x) \int d\nu(y)\, f(x,y) = \int d\nu(y) \int d\mu(x)\, f(x,y)$.

- OK if f is nonneg (Tonelli)
- OK if f is integrable for $\mu \otimes \nu$. (Fubini).

$\to$ allow "$\infty = \infty$" as a possibility.

  e.g. $f(x,y) = e^{-x^2} \dfrac{1}{\sqrt{1+y^2}}$ on $\mathbb{R}^2$

$\to$ only allow conv. integs. but allow $\mathbb{R}$-valued.

  (or $\mathbb{C}$-valued). e.g. $f(x,y) = \cos(x+y) e^{-(x^2+y^2)}$

Th'm (Tonelli ; second part of 20.5 / 20.8).

  $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$ σ-finite meas. sp.

  $f : X \times Y \to [0, \infty]$ $\mathcal{M} \otimes \mathcal{N}$-meas.

  Then $\int d\mu(x) \int d\nu(y)\, f(x,y) = \int d\nu(y) \int d\mu(x)\, f(x,y)$

Idea : follow same pattern as before

  Step 1. Assume $\mu(X)$, $\nu(Y) < \infty$,

  check claim for $f(x,y) = 1_A(x) 1_B(y)$

Step 2. Step 1 $\Rightarrow$ claim for $f = 1_E$
$E \in \mathcal{M} \otimes \mathcal{N}$.

Claim is compat. with taking
- complements
- countable union.

(formally use Dynkin system)

Step 3. arbitrary bounded f
(still $\mu(X), \nu(Y) < \infty$)

- simple funcs. approximate f from below
- partial integs are measurable (20.5)
- monotone convergence thm

Step 4 remove bddness assumptions on f, X, Y ☒

Thm (Fubini, 20.9).

Setting : $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$: σ-fin.

When does it apply? : $f : X \times Y \to \mathbb{R}$ $\mathcal{M}\otimes\mathcal{N}$-meas.

$$\int |f| \, d\mu\otimes\nu = \int d\mu(x) \int d\nu(y) \, |f(x,y)| = (x \leftrightarrow y, \mu \leftrightarrow \nu)$$

part of claim.

What does it say? : if these are finite :
- $f(x,y)$ is μ-integrable for a.e. y
- $x \leftrightarrow y$, $\mu \leftrightarrow \nu$.
- $\int f \, d\mu\otimes\nu = \int d\mu \int d\nu \, f = \int d\nu \int d\mu \, f$

Proof Step 1 $\int |f| \, d\mu\otimes\nu = \int d\mu d\nu \, |f|$
from Tonelli

Step 2 $f(x,y)$ is μ-int'ble for a.e. y.

Want (assuming $\int d\nu \int d\mu |f| < \infty$)

$$B_\infty = \{ y \in Y : \int |f(x,y)| d\mu(x) = \infty \}$$

satisfies $\nu(B_\infty) = 0$.

reason: $\infty \cdot \nu(B_\infty) \leq \int \big(\underbrace{\int |f(x,y)| d\mu(x)}_{\infty \text{ on } B_\infty,\ \text{nonneg. else}} \big) d\nu(y)$

must be 0.

Step 3. $\int f \, d\mu \otimes \nu = \int d\nu \int d\mu f$

Write $f = f^+ - f^-$ $\qquad 0 \leq f^\pm \leq |f|$.

pos part, neg part.

- $f^\pm$ satisfy this eq. by Tonelli
- int. of f = int of f^+ − int. of f^-

for $\int d\mu\otimes\nu$, $\int d\nu \int d\mu$ $\boxed{}$

A motivation:

Suppose $X = Y = \mathbb{N}$, $\mu = \nu$ counting measure.

$\rightsquigarrow$ - meas. func. $f : X \times Y \to \mathbb{R}$: double seq. $(f(m,n))_{m,n=1}^\infty$

- integration (for $\mu, \nu, \mu\otimes\nu$) summation $\sum_m f(m,n)$, etc.

$\sum_{m,n} |f(m,n)| < \infty$ means f is absolutely summable

$\Rightarrow$ we can enumerate pts of $\mathbb{N} \times \mathbb{N}$ as

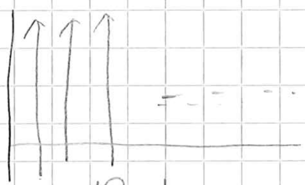
$(m_1, n_1), (m_2, n_2), \ldots$

and take $\sum_{k=1}^\infty f(m_k, n_k)$ this is indep. of order.

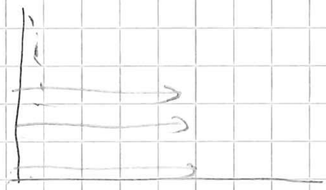
$\underline{\text{if}}$ $\sum |f(m,n)| = \infty$, but $f(m,n) \to 0$
and mixed signs

we can arrange ordering of (m_k, n_k)

so $\lim_{K \to \infty} \sum_{k=1}^{K} f(m_k, n_k)$ is quite arbitrary num



vs.



could have weird cancellation

if $\sum |f(m,n)| = \infty$

Variations

1. $(X_1, \mathcal{M}_1, \mu_1), \dots, (X_n, \mathcal{M}_n, \mu_n)$ σ-fin

$\rightsquigarrow \mu_1 \otimes \cdots \otimes \mu_n$ on $X_1 \times \cdots \times X_n$

by repeating $\mu_1 \otimes \cdots \otimes \mu_k, \mu_{k+1} \rightsquigarrow \mu_1 \otimes \cdots \otimes \mu_{k+1}$

2. sequence $\underbrace{(X_i, \mathcal{M}_i, \mu_i) \quad i = 1, 2, \dots}_{\mu_i(X_i) = 1}$

$\bigotimes_{i=1}^{\infty} \mu_i$ on $\prod_{i=1}^{\infty} X_i$

characterized by $\left(\bigotimes_{i=1}^{\infty} \mu_i\right)\left(\prod A_i\right) = \prod_{i=1}^{\infty} \underbrace{\mu_i(A_i)}_{0 \le \cdot \le 1}$

Without $\mu_i(X_i) = 1$ we need to take a "restricted product".

Examples

1. Lebesgue measure on $\mathbb{R}^d$: $m^d = \underbrace{m \otimes \cdots \otimes m}_{d \text{ times}}$

m : L-meas. on $\mathbb{R}$

2. probability measure of infinite coin toss

$\mu = \bigotimes_{i=1}^{\infty} \mu_0$ on $\prod_{i=1}^{\infty} \{0, 1\}$, $\mu_0(\{0\}) = \mu_0(\{1\}) = \frac{1}{2}$