Last time: multiple integrals

$\int d\mu(x) \int d\nu(y)\, f(x,y)$ vs. $\int d\nu(y) \int d\mu(x)\, f(x,y)$.

these are eq. when: (μ, ν: σ-fin., f meas.)

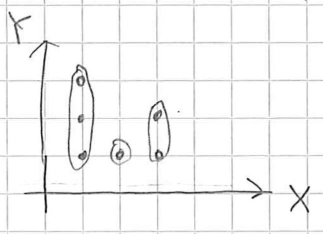
- $f \geq 0$ (Tonelli)
- f integrable for $\mu \otimes \nu$ (Fubini)

More illustrating example

$X = Y = \mathbb{N}$, $\mu = \nu$ = counting measure.

1. $\mu \otimes \nu$ is the counting measure on $\mathbb{N} \times \mathbb{N}$.

By def. $(\mu\otimes\nu)(E) = \int d\mu(x) \int d\nu(y)\, 1_E(x,y)$.



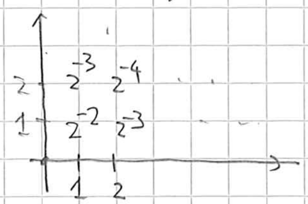
For fixed x,

$F(x) = \int d\nu(y)\, 1_E(x,y)$

is $\nu(E_x)$ for $E_x = \{y : (x,y) \in E\}$

$\Rightarrow (\mu\otimes\nu)(E) = \sum_{x=1}^{\infty} \underset{\uparrow \text{ count. meas.}}{\nu}(E_x) = |E|$.

2. integration of $f(x,y) = 2^{-(x+y)}$.



approximation by simple function

~ approx. f on a (large) rect.

2-1. $f_n(x,y) = \begin{cases} f(x,y) & x, y \leq n \\ 0 & x > n \text{ or } y > n. \end{cases}$

$$= \underbrace{\sum_{x=1}^{n} \sum_{y=1}^{n}}_{\text{finite sum}} \underbrace{2^{-(x+y)}}_{\text{coeff.}} \underbrace{1_{\{(x,y)\}}}_{1_{\{x\}\times\{y\}}} \text{ measurable}$$

then $f_1 \leq f_2 \leq \cdots$, $f(x,y) = \lim_{n\to\infty} f_n(x,y)$

2-2.. $\int d\mu(x) \int d\nu(y)\, f_n(x,y) = \int d\nu(y) \int d\mu(x)\, f_n(x,y)$

we are comparing $\sum_{x=1}^{n}\sum_{y=1}^{n} 2^{-(x+y)}$ vs. $\sum_{y=1}^{n}\sum_{x=1}^{n} 2^{-(x+y)}$

$\rightsquigarrow$ ok.

2-3 $\int f_n \, d\mu\otimes\nu$ = above iter. integs.

$$\sum_{(x,y)\in\{(1,1),\cdots(1,n),\cdots(n,n)\}} 2^{-(x+y)} = \sum_{x=1}^{n}\sum_{y=1}^{n} 2^{-(x+y)}.$$

2-4. $\int d\mu \int d\nu\, f = \int d\nu \int d\mu\, f$

Fixed x: $\underset{\sum_{y=1}^{\infty} 2^{-(x+y)}}{\int f(x,y)\, d\nu(y)} = \lim_{n\to\infty} \int f_n(x,y)\, d\nu(y)$

(because $0 \le f_n(x,y) \overset{\lim_n \sum_{y=1}^{n} 2^{-(x+y)}}{\nearrow} f(x,y)$ as func. in y

and "monotone conv. thm")

Put $F_n(x) = \int f_n(x,y)\, d\nu(y)$. So $F_n(x) \nearrow F(x)$.

$\int d\mu \underbrace{\int d\nu\, f}$ is $\sum_{x=1}^{\infty}\sum_{y=1}^{\infty} 2^{-(x+y)}$

$\lim_{n\to\infty} \int d\nu\, f_n$ by above.

Switching these: monotone conv. thm.

or $\underset{\lim_{m\to\infty}\sum_{x=1}^{m}}{\sum_{x=1}^{\infty}} \lim_{n\to\infty} \sum_{y=1}^{n} = \lim_{n\to\infty} \underset{\lim_{m\to\infty}\sum_{x=1}^{m}}{\sum_{x=1}^{\infty}} \sum_{y=1}^{n}$

$\uparrow$ switching limits

$\rightsquigarrow \int d\mu \int d\nu\, f = \lim_{n\to\infty} \int d\mu \int d\nu\, f_n$

$\int d\nu \int d\mu\, f = \lim_{n\to\infty} \int d\nu \int d\mu\, f_n$

How to "use" Fubini: justify manipulations

around multiple integs

Example.

Laplace transform of $\frac{\sin t}{t}$: $f(s) = \int_0^{\infty} \frac{\sin t}{t} e^{-st}\, dt.$

$s>0$

trick: $\frac{1}{t} = \int_0^{\infty} e^{-tx}\, dx.$

so $f(s) = \int_0^\infty dt \int_0^\infty dx \left((\sin t) \times e^{-tx} \times e^{-st} \right)$

$= \int_0^\infty dx \int_0^\infty dt \left((\sin t) \times e^{-t(x+s)} \right)$

used Fubini

$\sin t \times e^{-ta} = \frac{e^{it} - e^{-it}}{2i} e^{-ta}$

has prim. func. $\frac{1}{2i} \left(\frac{e^{(i-a)t}}{i-a} + \frac{e^{-(i+a)t}}{i+a} \right)$.

so $\int_0^\infty dt \left(\sin t \times e^{-t(x+s)} \right) = \frac{1}{1+(x+s)^2}$

overall $\int_0^\infty \frac{dx}{1+(x+s)^2} = \frac{\pi}{2} - \mathrm{Arctan}\, s$.

why could we use Fubini ?

$\sin t \times e^{-t(x+s)}$ was integrable on $[0,\infty)^2$ as func. in t, x.

$|\sin t \times e^{-t(x+s)}| \leq \underbrace{e^{-t(x+s)} \times (\min(1,t))}_{\text{non neg. fin. int.}}$

$\int_0^\infty e^{-t(x+s)} dx = \frac{e^{-ts}}{t}$

$\leadsto$ we need to estim. $\int_0^1 e^{-ts} dt + \int_1^\infty \frac{e^{-ts}}{t} dt$