Dynkin system

We used "Dynkin system" (Aug 25 Lem 20.2 Step 2
Aug 31 Tonelli's thm step 2) like this.

Setting : $(X, \mathcal{M})$, $(Y, \mathcal{N})$ meas. sp.

Want : $\mathcal{M} \otimes \mathcal{N}$ (or $f = 1_E$ for $E \in \mathcal{M} \otimes \mathcal{N}$)
has some "nice" property

Strategy : put $\mathcal{L} = \{ F \subset X \times Y \text{ nice} \}$

1. check $A \times B \in \mathcal{L}$ if $A \in \mathcal{M}$, $B \in \mathcal{N}$
2. check that $\mathcal{L}$ is a "Dynkin system"

Conclusion : $\mathcal{M} \otimes \mathcal{N}$ (σ-a's generated by the collection $\{ A \times B : A \in \mathcal{M}, B \in \mathcal{N} \}$)
is a subcollection of $\mathcal{L}$. ($\mathcal{M} \otimes \mathcal{N} \subset \mathcal{L}$)

What is a Dynkin system ?

Def (33.1) Z : set. $\mathcal{L}$: collection of subsets of Z.
$\mathcal{L}$ is a <u>Dynkin system</u> (or λ-system, λ-class)
if
1. $Z \in \mathcal{L}$
2. $A, B \in \mathcal{L}$, $A \supset B \Rightarrow A \setminus B \in \mathcal{L}$.
3. $A_1 \subset A_2 \subset \cdots$, $A_i \in \mathcal{L} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{L}$.

Rem. Another (equivalent) def. of D-system

1'. $Z \in \mathcal{L}$

2'. $A \in \mathcal{L} \Rightarrow A^c = X \setminus A \in \mathcal{L}$

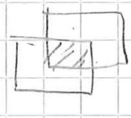
3'. $A_1, A_2, \cdots \in \mathcal{L}$, $A_i \cap A_j = \emptyset$ $(i \neq j)$
$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{L}$

3' : $\underbrace{A_1 \cup A_2}_{\text{disj.}} = \underbrace{(A_2^c \setminus A_1)^c}$ $\in \mathcal{L}$ by 2.

Def. $\mathcal{L}$ collection of subsets of Z is a <u>π-class</u> if $A_1, \dots, A_n \in \mathcal{L} \Rightarrow \bigcap_{i=1}^{n} A_i \in \mathcal{L}$.

Ex. $Z = X \times Y$, $\mathcal{L} = \{ A \times B : A \in \mathcal{M}, B \in \mathcal{N} \}$

$$\bigcap_{i=1}^{n} (A_i \times B_i) = \left(\bigcap_{i=1}^{n} A_i \right) \times \left(\bigcap_{i=1}^{n} B_i \right)$$



so $\mathcal{M}, \mathcal{N}$ σ-alg $\Rightarrow$ $\mathcal{L}$ is a π-class.

Rem σ-alg $\Rightarrow$ Dynkin sys.

Thm ("π-λ theorem" 33.5)

Z: set $\mathcal{C}, \mathcal{L}$: collections of subsets of Z.

$\mathcal{C}$: π-class, $\mathcal{L}$: Dynkin sys., $\mathcal{C} \subset \mathcal{L}$.

Then $\sigma(\mathcal{C}) \subset \mathcal{L}$

Idea: Step 1. replace $\mathcal{L}$ by smallest possible

(use $\mathcal{L}'$ = intersection of all D-sys. $\mathcal{L}''$ s.t. $\mathcal{C} \subset \mathcal{L}''$)

Step 2. $A \in \mathcal{L}$, $B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{L}$

Step 3 $A, B \in \mathcal{L} \Rightarrow A \cap B \in \mathcal{L}$

Step 4 $\mathcal{L}$ is a σ-alg.

Details Step 2: Put $\mathcal{D} = \{ A \in \mathcal{L} : \forall B \in \mathcal{C} : A \cap B \in \mathcal{L} \}$

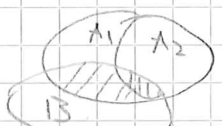
Want $\mathcal{D} = \mathcal{L}$.

Claim $\mathcal{D}$ is a Dynkin system. ($\Rightarrow \mathcal{D} = \mathcal{L}$ by $\mathcal{C} \subset \mathcal{D}$ and minimality of $\mathcal{L}$.)

cond. 1 $X \cap B = B \in \mathcal{C} \subset \mathcal{L}$

2 $A_1, A_2 \in \mathcal{D}$, $A_1 \supset A_2$ want $A_1 \setminus A_2 \in \mathcal{D}$

$$(A_1 \setminus A_2) \cap B = (\underbrace{A_1 \cap B}_{\text{in } \mathcal{L}}) \setminus (\underbrace{A_2 \cap B}_{\text{in } \mathcal{L}}) \in \mathcal{L} \quad \text{by 2 for } \mathcal{L}$$



3 $A_1 \subset A_2 \subset \cdots$, $A_i \in \mathcal{D}$ want $\bigcup A_i \in \mathcal{D}$

$$\left(\bigcup A_i\right) \cap B = \bigcup (\underbrace{A_i \cap B}_{\text{in } \mathcal{L}}) \in \mathcal{L} \quad \text{by 3 for } \mathcal{L}$$

Step 3 Again put $\mathcal{D}' = \{A \in \mathcal{L} : \forall B \in \mathcal{L} : A \cap B \in \mathcal{L}\}$

Want $\mathcal{D}' = \mathcal{L}$.

Claim $\mathcal{D}'$ is a Dynkin system.

similar args. as before

$\mathcal{C} \subset \mathcal{D}'$ by Step 2 $\underset{\text{minimality of } \mathcal{L}}{\Rightarrow}$ $\mathcal{D}' = \mathcal{L}$

Step 4. $\mathcal{L}$ is closed under

- complement
- finite intersection
- union of increasing sequence

$\Rightarrow$ σ-alg. $\boxed{}$

Turning back to Lem 20.2.

Setting $(X, \mathcal{M})$, $(Y, \mathcal{N})$ meas. sp.

$f : X \times Y \to \mathbb{R}$ $\mathcal{M} \otimes \mathcal{N}$-measurable

Goal : $f_y(x) = f(x, y)$ is $\mathcal{M}$-measurable.

"Step 2" $E \in \mathcal{M} \otimes \mathcal{N} \Rightarrow f = 1_E$ has this prop.

We put $\mathcal{L} = \{E \subset X \times Y : (1_E)_y \text{ is meas.}\}$

1' : $X \times Y \in \mathcal{L}$, 2' : $E \in \mathcal{L} \Rightarrow E^c \in \mathcal{L}$.

$1_{E^c} = 1 - 1_E$

3' : $A_1, A_2, \cdots \in \mathcal{L}$, $A_i \cap A_j = \emptyset$ $(i \neq j)$

(don't need here)

$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{L}$ $\qquad 1_{\bigcup_{i=1}^{\infty} A_i} = \sup_{i=1,2,\cdots} 1_{A_i}$

$\mathcal{C} = \{A \times B : A \in \mathcal{M}, B \in \mathcal{N}\} \underset{\text{"Step 1"}}{\subset} \mathcal{L}$, π-sys.

$\Rightarrow \sigma(\mathcal{C}) \subset \mathcal{L}$ by π-λ thm.

might not be equality