Exercise problem 2 (set 1)

We'll use

- Tonelli's theorem : to compare iterated integrals and integrals for product measure
  of nonneg. funcs
- "change of variables" formula for integrals of Lebesgues measure (on plane)

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

$$\Gamma(x)\Gamma(y) = \int_0^\infty dt \int_0^\infty ds \left(t^{x-1} e^{-t} \times s^{y-1} e^{-s}\right)$$

$$\overset{\text{Tonelli}}{=} \int_{(0,\infty)\times(0,\infty)} t^{x-1} s^{y-1} e^{-(t+s)} \, dm^2(t,s)$$

↑ Lebesgue meas. on $\mathbb{R}^2$.

Consider change of variables $(t,s) = (uv, u(1-v))$

$0 < v < 1$, $0 < u < \infty$. $\left(t+s = u, \frac{t}{t+s} = v\right)$

the above integral is equal to

$$\int_{(0,\infty)\times(0,1)} (uv)^{x-1} (u(1-v))^{y-1} e^{-u} \left| \underbrace{\det \begin{pmatrix} \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \\ \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \end{pmatrix}}_{\text{write } JT} \right| dm^2(u,v)$$

$$JT = \begin{pmatrix} u & 1-v \\ u & -u \end{pmatrix} \Rightarrow |\det JT| = u.$$

so we have

$$\int u^{x+y-2+1} e^{-u} v^{x-1} (1-v)^{y-1} \, dm^2(u,v)$$

$$\overset{\text{Tonelli}}{=} \int_0^\infty du \int_0^1 dv \text{ (above integrand)}$$

$$= \underbrace{\int_0^\infty u^{x+y-1} e^{-u} du}_{\Gamma(x+y)} \times \int_0^1 v^{x-1} (1-v)^{y-1} dv$$

"change of variable formula"

we need $\int f(t,s)\, dm^2(t,s) = \int f\circ T(u,v)\, |\det JT|\, dm^2(u,v)$

for - nonneg. measurable func. $f(t,s)$

- $T: (0,\infty)\times(0,1) \to (0,\infty)\times(0,\infty)$

$(u,v) \longmapsto (uv,\ u(1-v))$

observation : $T = T''\circ T'$ with

$T'(u,v) = (u, s(u,v))$ for $s(u,v) = u(1-v)$

$T''(u,s) = (t(u,s), s)$ for $t(u,s) = u-s$

$$\left(\begin{array}{l} T' : (0,\infty)\times(0,1) \to \{(u,s) : u>s>0\} \\ T'' : \{(u,s) : u>s>0\} \to (0,\infty)\times(0,\infty) \end{array}\right.$$

we have JT = "$JT''\circ JT'$" so it's enough

to establish $\int f(t,s)\, dm^2(t,s) = \int (f\circ T'') \times |\det JT''|\, dm^2(u,s)$

and $\int \tilde{f}\, dm^2(u,s) = \int (\tilde{f}\circ T') \times |\det JT'|\, dm^2(u,v)$

separately We'll check the second

$$JT' = \begin{pmatrix} \frac{\partial u}{\partial u} & \frac{\partial u}{\partial v} \\ \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1-v & -u \end{pmatrix} \to |\det JT'| = u$$

$$\int \tilde{f}(u, s(u,v)) \times u\, dm^2(u,v)$$

$$\overset{\text{Tonelli}}{=} \int dm(u) \int dm(v)\ \tilde{f}(u, s(u,v))\ \underset{\frac{\partial s}{\partial v}}{\underset{\uparrow}{u}}$$

$\frac{\partial s}{\partial v}$ up to sign.

$$= \int dm(u) \left(\int_{\substack{v=0 \\ s=u}}^{\substack{s=0 \\ v=1}} \tilde{f}(u,s) \left(-\frac{\partial s}{\partial v}\right) dv \right)$$

$$= \int dm(u) \left(\int_{s=0}^{s=u} \tilde{f}(u,s)\, ds \right) = \int dm(u) \int dm(s)\ \tilde{f}(u,s)$$

$$\overset{\text{Tonelli}}{=} \int \tilde{f}(u,s)\, dm^2(u,s)$$