Banach spaces basics.

Setting : normed vector space

- vector space $V \ni X$, coefficients $\mathbb{R}$ or $\mathbb{C}$
- norm $\|v\|$ (x) for $v \in V$
  - $\|v\| \geq 0$, $\|v\| = 0 \iff v = 0$
  - $\|u + v\| \leq \|u\| + \|v\|$ triangle ineq.
  - $\|c v\| = |c| \cdot \|v\|$ for coeff. c

$\leadsto$ metric (distance) $d(u, v) = \|u - v\|$.

Def. (7.1) A Banach space is a normed vec. sp. V that is complete as a metric sp. i.e. if $v_1, v_2, \cdots$ is a Cauchy seq. in V, $\exists v_\infty \in V$ s.t. $\|v_n - v_\infty\| \to 0$ $(n \to \infty)$

Rem completion

Rem. we'll work with real coeffs most of the time

Examples

1. $L^p(X, \mathcal{M}, \mu)$ (Y, N, ν) for measure space $(X, \mathcal{M}, \mu)$ $1 \leq p \leq \infty$.

   triangle ineq. for L^p $(1 \leq p < \infty)$ is called Minkowski's inequality (conseq. of Hölder's ineq.

   for $p = \infty$: conseq. of $|f(x) + g(x)| \leq |f(x)| + |g(x)|$

   completenss of L^p : use convergence in measure, etc.

2. Sobolev spaces

   e.g. $W^{k,p}(\mathbb{R}) = \{ f \in L^p(\mathbb{R}),\ \text{"}\int \left|\frac{d^j f}{dx^j}\right|^p dx < \infty\ \forall j \leq k\text{"} \}$

   with norm $\left(\|f\|_p^p + \|f'\|_p^p + \cdots + \|f^{(k)}\|_p^p \right)^{1/p}$

   $= \left(\int |f|^p + \cdots + \left|\frac{d^k f}{dx^k}\right|^p dx \right)^{1/p}$

big p : assume better local regularity

small p : assume stronger decay at ∞.

Linear transforms

Def (7.7) X, Y normed vector spaces

A <u>bounded</u> linear map (bdd. lin. transform) :

linear map $T : X \to Y$ s.t. $\exists\, C > 0$ s.t.

$$\|Tx\| \le C\|x\|.$$

The norm of T : optimal const C as above

$$\|T\| = \inf C : \forall x \ \|Tx\| \le C\|x\|$$

$$= \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\| \le 1} \|Tx\|$$

Def (7.9) $L(X,Y) = \{T : \text{bdd lin. map } X \to Y\}$

norm as above.

$X^* = L(X, \mathbb{R})$: <u>dual</u> space of X.

($X^* = L(X, \mathbb{C})$ for complex coeff.)

(bounded linear) <u>functionals</u> on X :

elements of X^* ; norm $\|\phi\|_{L(X,\mathbb{R})} = \|\phi\|_{X^*}$

Example $p, q \geq 1$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$ (conjugate ind.)

$\|f \cdot g\|_1 \le \|f\|_p \cdot \|g\|_q$ for measurable funcs

on $(X, \mathcal{M}, \mu)$. (<u>Hölder's</u> ineq.)

This means :

- $g \in L^q(X,\mu)$ defines a functional ϕ_g on $L^p(X,\mu)$: $\phi_g(f) = \int f(x) g(x)\, d\mu(x)$
- $\|\phi_g\|_{L^p(X,\mu)^*} \le \|g\|_q$