Converse of Hölder's inequality (§21.4)

Hölder's ineq. said $\|f\cdot g\|_1 \le \|f\|_p \cdot \|g\|_q$ for $\frac{1}{p}+\frac{1}{q}=1$

($\Leftrightarrow$ $\|g\|_q \le \|\phi_g\|_{(L^p)^*}$ see prev lecture)

when $p=2$ $(q=2)$: $f(x)=g(x)$ achieves

$\|f\cdot g\|_1 = \|f\|_2 \cdot \|g\|_2 \to |\int |g|^2 d\mu|^{1/2} \times |\int |g|^2 d\mu|^{1/2}$

$\hookrightarrow \int |g\, g|\, d\mu = \int |g|^2 d\mu$

Prop (21.26) $(X, \mathcal{M}, \mu)$: σ-finite meas. sp. (only need for $p=1$)

$1 \le p \le \infty$, $\frac{1}{p}+\frac{1}{q}=1$ (when $p=1$ set $q=\infty$, etc.)

$g \in L^q(X, \mathcal{M}, \mu)$, $\varepsilon > 0$

Then $\exists f \in L^p(X, \mathcal{M}, \mu)$ s.t. $\varepsilon + \|f\cdot g\|_1 \ge \|f\|_p \cdot \|g\|_q$.

($\Rightarrow$ $\|g\|_q = \|\phi_g\|_{(L^p)^*}$)

Idea : for $p=2$ we took $f(x)=g(x)=\mathrm{sgn}(g(x))\,|g(x)|$

$p \ne 2$: we have to take fractional powers of $|g(x)|$

want : $|f(x)|^p = |g(x)|^q$

so $\|f\|_p \cdot \|g\|_q = \left(\int |g(x)|^q d\mu\right)^{1/p} \left(\int |g(x)|^q d\mu\right)^{1/q} = \|g\|_q^q$

edge cases : $p=1$, $q=\infty$ or $p=\infty$, $q=1$.

$p=\infty$: take $f(x)=\mathrm{sgn}(g(x))$ (± 1)-valued

$p=1$ (only time we need σ-finiteness)

fix $A_1 \subset A_2 \subset \cdots$ measurable, $\mu(A_i)<\infty$, $X=\bigcup_{i=1}^{\infty} A_i$

Put $M = \|g\|_\infty$, $B_n = \{x \in A_n : |g(x)| \ge M-\varepsilon\}$

Claim. $\exists i$ s.t. $\mu(B_i) > 0$

Proof of claim : $B_n \nearrow \underbrace{\{x\in X : |g(x)| \ge M-\varepsilon\}}_{\text{has pos. meas. by } \|g\|_\infty = M.}$

Take i as in claim, put $f' = \mathrm{sgn}(g)\cdot 1_{B_i}$

then $\|f'\cdot g\|_1 = \int_{B_i} |g|\, d\mu \ge (M-\varepsilon)\,\mu(B_i)$

So $\|f'\cdot g\|_1 \geq (\|g\|_\infty - \varepsilon)\cdot \underbrace{\mu(B_i)}_{\|f'\|_1}$

$f(x) = \frac{1}{\mu(B_i)} f'(x) = \frac{\mathrm{sgn}(g(x))}{\mu(B_i)} \cdot 1_{B_i}(x)$ satisfies $\|f\|_1 = 1$.

$$\|f\cdot g\|_1 \geq \|g\|_\infty \cdot \|f\|_1 - \varepsilon . \qquad \square$$

Rem. analogue for complex-valued functions: replace "$\mathrm{sgn}(g)$" by $\overline{g(x)}$.

Summary: $g \mapsto \phi_g$ is an isometric embedding of $L^q(X,\mu)$ into $L^p(X,\mu)^*$.

Goal (later) this is also surjective.

Density (§22.3)

What kind of functions can approximate L^p-funcs?

Lem. (22.3) $1 \leq p < \infty$. $S_f(X, M, \mu) = \{ \sum_{i=1}^{\infty} t_i 1_{B_i} : B_i \in M, \ \mu(B_i) < \infty \}$ is dense in $L^p(X, M, \mu)$

Proof: fix $f \in L^p(X,\mu)$; want to approximate f by $\phi_1, \phi_2, \dots \in S_f(X,\mu)$ (in L^p norm)

Step 1 reduction to $f \geq 0$.

decompose $f = f^+ - f^-$, $f^\pm \geq 0$.

Step 2 take nonneg. simple funcs $\phi_1 \leq \phi_2 \leq \dots \nearrow f$.

(without asking $\phi_n \in S_f$)

Step 3 check $\phi_n \in S_f$: $\underbrace{|\phi_n|^p}_{\sum t_i^p 1_{B_i} \text{ if } \phi_n = \sum t_i 1_{B_i}} \leq |f|^p$ integrable.

Step 4 estimate $\|f - \phi_n\|_p$

$h_n = |f - \phi_n|^p \leq (|f| + |\phi_n|)^p \leq 2^p |f|^p$

$\rightsquigarrow$ use dominated conv. thm for h_n.

$h_n(x) \to 0$ $(n \to \infty)$ gives $\int h_n \, d\mu \to 0$.

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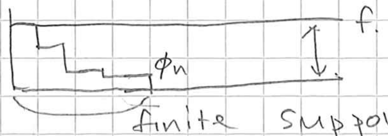
Rem. $S_f(X,\mu)$ is <u>not</u> dense in $L^\infty(X,\mu)$ if $\mu(X)=\infty$. ; cost. funcs can't be approxed



Other examples. (w/o proof for now)

1. $(X,\mu)=(\mathbb{R}, m)$
   Lebesgue meas.

$C_c(\mathbb{R}) = \{ f : \mathbb{R}\to\mathbb{R}$ cont, $\mathrm{supp}(f)$ cpt $\}$

$C_c^\infty(\mathbb{R}) = \{ f$: as above, $f', f'', \dots$ exist$\}$

are dense in $L^p(\mathbb{R}, m)$, but not in $L^\infty(\mathbb{R}, m)$

2. probability space $(\mu(X)=1)$

$L^\infty(X,\mu)$ is dense in $L^p(X,\mu)$