Digression : why L^p ?

- $p=2$: $L^2(X,\mu)$ is a Hilbert space
  $\rightsquigarrow$ good framework to understand eigenproblems.
  (particularly over complex coeff.)
- $p=1$: integrable functions
  density of finite measure, ...
- $p=\infty$: bounded functions.
  closed under product, ...

Intermediate value of p : useful to understand nonlinear equations.

Example $\quad -\Delta(u(x)+u_0(x)) = u^2(x) \qquad x \in \mathbb{R}^n$; $u_0(x)$ fixed.

Ansatz : express it as a fixed point problem

$\Phi(u) = u \qquad$ for $\quad \Phi(u) = (-\Delta)^{-1}(u^2) - u_0$

Find a complete metric space (X, d) (of functions) s.t. Φ acts as contraction $\quad d(\Phi(u), \Phi(v)) \le d(u,v)$

Fact (Sobolev embedding) $\quad W^{k,p}(\mathbb{R}^n) \subset W^{\ell,q}(\mathbb{R}^n)$

$$\frac{1}{p} - \frac{k}{n} = \frac{1}{q} - \frac{\ell}{n}.$$

In particular $\quad \|g\|_{L^q(\mathbb{R}^n)} \le C \|-\Delta g\|_{L^p(\mathbb{R}^n)}$

for $\quad \frac{1}{p} - \frac{2}{n} = \frac{1}{q} \qquad (\ell = 0, k = 2)$

Combine this with $\|u^2\|_p = \|u\|_{2p}^2$

$$\|\Phi(u)\|_q \le C\|u\|_{2p}^2 + \|u_0\|_q$$

When $n=3$: the choice $q=\frac{3}{2}$ achieves $q=2p$.

($\rightsquigarrow$ we can take $L^{\frac{3}{2}}(\mathbb{R}^3)_{R_0}$ as the compl. met. space where Φ acts.

(NB intermediate steps involve $\|f\|_{\frac{3}{4}}$!)

With a bit more work $\|u\|_{\frac{3}{2}}, \|v\|_{\frac{3}{2}} < R$

$$\Rightarrow \|\Phi(u) - \Phi(v)\|_{\frac{3}{2}} \leq 2RC\|u-v\|_{\frac{3}{2}}.$$

So Φ is contractive on a R_0-ball $(R_0 \leq \frac{1}{2C})$

if u_0 is also small in $\frac{3}{2}$-norm.

Signed measure (§ 24).

Def (24.1) $(X, \mathcal{M})$ measurable space.

A <u>signed</u> <u>measure</u> on X is given by a map $\nu : \mathcal{M} \to [-\infty, \infty]$ s.t.

1. the range of ν does not contain both $\pm\infty$.
2. $A_1, A_2, \cdots \in \mathcal{M}$ mutually disj.

   then $\nu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \nu(A_i)$
3. $\nu(\emptyset) = 0$

Rem: In 2., $|\nu(\bigcup_{i=1}^{\infty} A_i)| < \infty \Rightarrow \sum_{i=1}^{\infty} |\nu(A_i)| < \infty$

from assumption 1.

Def (24.1) <u>σ-finiteness</u> : $X = \bigcup_{i=1}^{\infty} A_i$, $|\nu(A_i)| < \infty$.

<u>complex</u> <u>measure</u> : $\nu : \mathcal{M} \to \mathbb{C}$ (no "∞")

Example $(X, \mathcal{M}, \mu)$ meas, $f : X \to \mathbb{R}$, f^+ or f^- integrable

$$\nu(A) = \int_A f \, d\mu$$

Goal : integrable representation of signed meas.

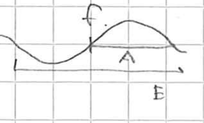
$\nu(A) = \int_A f \, d\mu$ under reasonable assumptions on ν.

"$d\nu = f \, d\mu$"

($\mu(A) = 0 \Rightarrow \nu(A) = 0$ ...)

Def. (24.5) $E \subset X$ is a null set for ν if

$A \in \mathcal{M}$, $A \subset E \Rightarrow \nu(A) = 0$.

! $d\nu =$  $. dm \Rightarrow \nu(E) = 0$ but $\nu(A) > 0$.

Th'm (baby Radon-Nikodym 24.6)

$(X, \mathcal{M}, \mu)$ finite meas. sp.

ν : signed meas. on X, $|\nu(A)| \le \mu(A)$.

Then $\nu(A) = \int_A \rho \, d\mu$ for some $\rho(x)$, $|\rho| \le 1$

Idea : - Make sense of $\int f \, d\nu$.

- Get ρ by the Riesz representation theorem on $L^2(X, \mu)$.

Step 1. $\int f \, d\nu$, for simple funcs.

$f = \sum_{i=1}^{n} t_i 1_{A_i} \Rightarrow \int f \, d\nu = \sum t_i \nu(A_i)$.

Step 2 extension to $L^2(X, \mu)$

- {simple funcs} $= \mathcal{S}_f(X, \mu)$ is dense in $L^2(X, \mu)$
$(\mu(X) < \infty)$

- $\left| \int f \, d\nu \right| \le \sqrt{\mu(X)} \, \|f\|_{L^2(X,\mu)}$ for simple funcs.

$\|f\|_{L^2(X,\mu)} = \sqrt{\sum_{i=1}^{n} t_i^2 \mu(A_i)} = \|u\|_2$

for $u := (t_i \sqrt{\mu(A_i)})_{i=1}^{n}$

with $v = (\sqrt{\mu(A_i)})_{i=1}^{n}$ $\qquad (u, v) = \sum t_i \mu(A_i) \ge \left|\int f \, d\nu\right|$

$\|v\|_2 \le \sqrt{\mu(X)}$

$|(u, v)| \le \|u\|_2 \cdot \|v\|_2$ implies claim

Step 3. Getting $p \in L^2(X, \mu)$.

by Step 2. $\int f \, d\nu$ makes sense as a

bdd lin. form on the Hilbert sp. $L^2(X, \mu)$

$\underset{\text{Riesz}}{\Rightarrow}$ $\exists p \in L^2(X, \mu)$ $\quad (f, p) = \int f \, d\nu$.

Step 4 $\quad \int_A p \, d\mu = \nu(A)$.

Take $f = \mathbb{1}_A$.