Exercise problem 1

$f(t)$: integrable func. on $0 < t < a$, $\quad g(x) = \int_x^a \frac{f(t)}{t}\,dt$

Integrability of g : $\int_0^a |g(x)|\,dx < \infty$

Claim : $|g(x)| \leq \int_x^a \frac{|f(t)|}{t}\,dt < \infty$

this follows from Minkowski's inequality for integrals

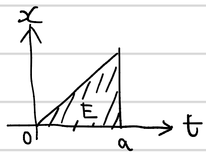
$$\left|\int h\,d\mu\right| \leq \int |h|\,d\mu$$

and $\frac{|f(t)|}{t} \leq \frac{|f(t)|}{x}$ on $x < t < a$

has finite integral in t in range

Then we have $\int_0^a |g(x)|\,dx \leq \int_0^a dx \int_x^a dt\, \frac{|f(t)|}{t}$

Put $E = \{(t,x) : 0 < x < t < a\} \subset (0,a)^2$



Then $\int_x^a \frac{|f(t)|}{t}\,dt = \int_0^a 1_E(t,x)\,\frac{|f(t)|}{t}\,dt$ for all x

So $\int_0^a dx \int_x^a dt\, \frac{|f(t)|}{t} = \int_0^a dx \int_0^a dt\; 1_E(t,x)\,\frac{|f(t)|}{t}$

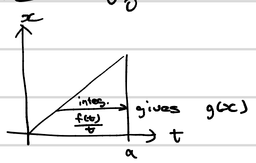
$$= \underbrace{\int_{(0,a)^2}}_{\text{Tonelli's th'm}} 1_E(t,x)\,\frac{|f(t)|}{t}\,\underbrace{dm^2(t,x)}_{\text{Lebesgues meas.}} = \underbrace{\int_0^a dt \int_0^a dx\; 1_E(t,x)\,\frac{|f(t)|}{t}}_{(*)}$$

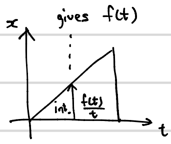
Now $\int_0^a 1_E(t,x)\,\frac{|f(t)|}{t}\,dx = \int_0^t \frac{|f(t)|}{t}\,dx = \frac{|f(t)|}{t} \times t = |f(t)|$ for any t

So $(*)$ is equal to $\int_0^a |f(t)|\,dt < \infty$ and we got $\int_0^a |g(x)|\,dx < \infty$

The equality $\int_0^a g(x)\,dx = \int_0^a f(t)\,dt$

Again similar idea!





We know $g(x)$ is integrable for the Lebesgues measure on $(0,a)$

Claim $1_E(t,x)\,\frac{f(t)}{t}$ is integrable on $(0,a)^2$

$\int_{(0,a)^2} \left|1_E(t,x)\,\frac{f(t)}{t}\right| dm^2(t,x) = \int_0^a |f(t)|\,dt$ by the above computation

(continued)

Then Fubini's th'm implies $\underbrace{\int_0^a dx \int_0^a dt \; 1_E(t,x) \frac{f(t)}{t}}_{\int_0^a g(x)\,dx} = \underbrace{\int_0^a dt \int_0^a dx \; 1_E(t,x) \frac{f(t)}{t}}_{\int_0^a f(t)\,dt}$