Def (24.7) $(X, \mathcal{M})$ measurable space
μ, ν : signed measures on X

1. μ and ν are <u>mutually singular</u> (write $\mu \perp \nu$)
   if $\exists A \subset X$ measurable s.t.
   - A is μ-null ($\Leftrightarrow \forall B \subset A$ meas. $\mu(B) = 0$)
   - $A^c = X \setminus A$ is ν-null
2. ν is <u>absolutely continuous relative to</u> μ
   (write $\nu \ll \mu$) if $\mu(A) = 0 \Rightarrow \nu(A) = 0$

Example. 1. $X = \mathbb{R}$ $\mu = m$, $\nu = \delta_0$. Dirac meas
$A = \{0\} \subset X$ is μ-null ($B \subset A \Rightarrow B = \{0\}$ or $B = \emptyset$)
$A^c = \mathbb{R} \setminus \{0\}$ is ν-null ($B \subset A^c \Rightarrow 0 \notin B$).

2. μ positive (i.e. usual) measure on X.
$\rho \in L^1(X, \mu)$, $\nu(A) = \int_A \rho \, d\mu \Rightarrow \nu \ll \mu$

Def. (24.9) $(X, \mathcal{M}, \mu)$ measure space
μ: usual meas.
ν : signed measure on X
A <u>Lebesgues decomposition</u> of ν relative to μ:
given by signed measures ν_a, ν_s on X,
1. $\nu(A) = \nu_a(A) + \nu_s(A)$ ...
2. $\nu_a \ll \mu$, $\nu_s \perp \mu$

Ex. $X = \mathbb{R}$, $\mu = m$
$$\nu(A) = \underbrace{\int_A e^{-x^2} \sin x \, dm(x)}_{\nu_a(A)} + \underbrace{\delta_0(A)}_{\nu_s(A)}$$

Lem (24.10) $(X, \mathcal{M}, \mu)$ meas. sp.
ν signed measure on X
0. Lebesgue dec. of ν (rel. to μ) is unique.
1. ν pos. $\Rightarrow \nu_a, \nu_s$ pos.

2. ν σ-fin. $\Rightarrow$ ν_a, ν_s σ-fin.

Proof. Fix $A \in X$ meas. s.t. $\mu(A)=0$ & A^c is ν_s-null

Step 1. $\forall\ C \in \mathcal{M}$ $\quad \nu_s(C) = \nu_s(C\cap A) = \nu(C\cap A)$

$\nu_a(C) = \nu_a(C\cap A^c) = \nu(C\cap A^c)$

$\because$ $C = (C\cap A) \sqcup (C \cap A^c)$

- $\nu_a \ll \mu$, $\mu(C\cap A) = 0 \Rightarrow \nu_a(C\cap A) = 0$
- A^c is ν_s-null. $\Rightarrow$ $\nu_s(C\cap A^c) = 0$

$\nu = \nu_a + \nu_s$ implies

$$\nu(C\cap A) = \underbrace{\nu_a(C\cap A)}_{} + \nu_s(C\cap A)$$

$$\nu_s(C) = \nu_s(C\cap A) + \underbrace{\nu_s(C\cap A^c)}_{0}$$

Step 2 uniqueness (claim 0)

take another decomp $\nu = \nu_a' + \nu_s'$ rel. to μ.

Step 1 implies $\nu_s'(C) = \nu(C\cap A) = \nu_s(C)$

$\nu_a'(C) = \nu(C\cap A^c) = \nu_a(C)$

Step 3. positivity (claim 1).

again from Step 1 $\nu_s(C) = \nu(C\cap A) \geq 0$, etc.

Step 4 σ-finiteness (claim 2)

Take $A_1 \subset A_2 \subset \cdots \subset X$ meas, $X = \bigcup_{i=1}^{\infty} A_i$

$|\nu(A_i)| < \infty$.

Then $\nu(A_i) = \nu_a(A_i) + \nu_s(A_i)$ implies

$\nu_a(A_i), \nu_s(A_i) \in \mathbb{R}$.

Th'm (existence of Lebesgue decomp & "the" Radon-Nikodym th'm 24.13)

$(X, \mathcal{M})$: meas. sp, μ, ν : σ-finite measures.

1. $\exists$ Lebesgue dec. $\nu = \nu_a + \nu_s$ rel. to μ.

2. $\exists\, p : X \to [0,\infty)$ meas. s.t. $d\nu_a = p\, d\mu$

Idea : reduce to the case when μ, ν finite.

then take $\lambda = \mu + \nu$, use "baby Radon-Nikodym" for λ and ν. $\Rightarrow$ $d\nu = h\, d\lambda$ for $0 \le h \le 1$.

$\leadsto$ $A = \{x : h(x) = 1\}$ is invisible from μ.

"$d\nu(x) = d\lambda(x)$"

$p = 1_{A^c} \cdot \dfrac{h}{1-h}$ will satisfy

$d\nu_a = p\, d\mu$. ("$h(d\nu_a + d\mu) = \frac{h}{1-h} d\mu$")

$\nu_s(C) = \nu(C \cap A)$.

Proof Step 1. The case $\mu(X), \nu(X) < \infty$.

Take λ, h, A, p as above.

- $\mu(A) = 0$:

$$\nu(A) = \int_A 1\, d\nu \underset{d\nu = h\, d\lambda}{=} \int_A h\, d\lambda \underset{h=1 \text{ on } A}{=} \mu(A) + \nu(A)$$

- $\nu_s(C) \underset{\text{def.}}{=} \nu(C \cap A)$
  $\nu_a(C) \underset{\text{def}}{=} \nu(C \cap A^c)$.
  $\Rightarrow$ $\left(\begin{array}{l} \mu \perp \nu_s. \\ \int_C (1-h)\, d\nu_a = \int_C h\, d\mu \end{array} \right.$ $\forall C$.

$(*)$: $\nu_a(C) = \int_{C \cap A^c} 1 . d\nu \overset{*}{=} \int_{C \cap A^c} h\, d\lambda = \int_{C \cap A^c} h\, d\nu + \int_C h\, d\mu$.

$$\Rightarrow \int_C (1-h)\, d\nu_a = \int_{C \cap A^c} (1-h)\, d\nu = \int_C h\, d\mu.$$

Step 2. $\int f (1-h)\, d\nu_a = \int f h\, d\mu$ $\left(\Rightarrow d\nu_a = \frac{h}{1-h} d\mu\right)$

for $f \geq 0$ meas. $(\Rightarrow \nu_a \ll \mu)$

$\therefore$ Step 1 $\Rightarrow$ claim for simple $f \geq 0$.

$\underset{\text{ptwise approx.}}{\Rightarrow}$ all $f \geq 0$.

Step 3 σ-finite case: reduction to Steps 1, 2

$Y_1 \subset Y_2 \subset \dots$ $\mu(Y_n) < \infty$, $X = \bigcup_{n=1}^{\infty} Y_n$

$Z_1 \subset Z_2 \subset \dots$ $\nu(Z_n) < \infty$, $X = \bigcup Z_n$

$\Rightarrow X'_n = Y_n \cap Z_n$ satisfies $\mu(X'_n), \nu(X'_n) < \infty$,

$X = \bigcup_{n=1}^{\infty} X'_n$

$\Rightarrow X_n = X'_n \setminus X'_{n-1}$ gives mut. disj. with same prop.

$\mu^n(C) = \mu(C \cap X_n)$, $\nu^n(C) = \nu(C \cap X_n)$

are finite measures, $\mu = \sum \mu^n$, etc.

Steps 1 & 2 give $\nu^n = \nu^n_a + \nu^n_c$ rel. to μ^n

"supported" on X_n

$\nu_a = \sum \nu^n_a$, $\nu_s = \sum \nu^n_s$ will do $\boxed{}$