L^p - L^q - duality

Recall: $\phi_g(f) = \int f(x)\, g(x)\, d\mu(x)$

- $|\phi_g(f)| \leq \|f \cdot g\|_1 \leq \|f\|_p \cdot \|g\|_q$ if $\frac{1}{p} + \frac{1}{q} = 1$ ← Hölder's inequality

- $\|g\|_q = \|\phi_g\|_{(L^p)^*}$ (so $L^q(X,\mu) \overset{\phi}{\underset{\text{isomet.}}{\hookrightarrow}} L^p(X,\mu)^*$)

Goal: ϕ is surjective, so $L^q(X,\mu)$ can be identified with $L^p(X,\mu)^*$ as a Banach space

What we need

- $1 \leq p < \infty$ ($L^\infty(X,\mu)^*$ is bigger than $L^1(X,\mu)$)
- (X, μ) is σ-finite.

($p = 2$ ($\Leftrightarrow q = 2$) OK since $L^2(X,\mu)$ is Hilb. sp.
use Riesz representation th'm

"Moral" reason for $1 \leq p \leq 2$, $\mu(X) < \infty$

$$\mu(X) < \infty \Rightarrow L^{p'}(X,\mu) \subset L^p(X,\mu) \quad \text{for } p' \geq p$$

$$\int |f|^p\, d\mu \leq \int_{\{x : |f(x)| \geq 1\}} |f(x)|^{p'}\, d\mu + \underbrace{\int 1\, d\mu}_{\text{finite}}$$

← compl.

If $\psi \in L^p(X,\mu)^*$, $\psi|_{L^2(X,\mu)}$ is cont. for $\|\sim\|_2$ norm $\Rightarrow \exists\, g \in L^2(X,\mu)$ s.t. $\psi(f) = \phi_g(f)$ $f \in L^2(X,\mu)$.

$L^2(X,\mu) \subset L^p(X,\mu)$ dense, ϕ_g is bdd (on L^2) in $\|\sim\|_p$ - norm $\underset{\text{"converse to Hölder"}}{\Rightarrow} \|g\|_q < \infty$

Th'm (24.14) (X, M, μ) σ-fin. meas. sp., $1 \leq p < \infty$
$\psi \in L^p(X,\mu)^*$ Then $\exists\, g \in L^q(X,\mu)$ $\frac{1}{p} + \frac{1}{q} = 1$
s.t. $\psi = \phi_g$

Proof we first assume $\mu(X) < \infty$. $(\Rightarrow L^\infty(X,\mu) \subset L^p(X,\mu))$

Step 1. $\nu(A) = \varphi(1_A)$ defines a signed measure on $(X, \mathcal{M})$

We need: If $A_1, A_2, \cdots \in \mathcal{M}$ are disjoint
$$\nu\left(\bigcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty \nu(A_i)$$

Claim $\|1_A - \sum_{i=1}^N 1_{A_i}\|_p \to 0 \quad (N \to \infty)$ for $A = \bigcup_{i=1}^\infty A_i$

$\because \|1_A - \sum_{i=1}^N 1_{A_i}\|_p = (\int |1_{B_N}|^p d\mu)^{1/p}$ $\quad B_N = \bigcup_{i=N+1}^\infty A_i$

$= \mu(B_N)^{1/p}$ (picture)

$\mu(A) < \infty$ and $\mu(A) = \sum_{i=1}^\infty \mu(A_i) \Rightarrow \mu(B_N) \xrightarrow{N\to\infty} 0$.

From claim $\underset{\nu(A)}{\underset{\parallel}{\varphi(1_A)}} = \lim \varphi(\sum_{i=1}^N 1_{A_i}) = \sum_{i=1}^\infty \varphi(1_{A_i})$
$$= \sum_{i=1}^\infty \nu(A_i).$$

Step 2 "total variation" $|\nu|(A) = \sup_{|f|\le 1} |\varphi(f 1_A)|$

is a (usual) measure, $|\nu|(A) \le \|\varphi\| \cdot \underbrace{\mu(A)^{1/p}}_{\|1_A\|_{L^p(X,\mu)}}$.

$A_1, A_2, \cdots \in \mathcal{M}$ mutually disj.

$\Rightarrow |\nu|(\bigcup_{i=1}^\infty A_i) \le \sum_{i=1}^\infty |\nu|(A_i)$ from $\|f \cdot \sum_{i=1}^N 1_{A_i}\|_p \le \mu(B_N)^{1/p} \to 0$. $(|f| \le 1)$

A, B disj $\Rightarrow |\nu|(A \cup B) \ge |\nu|(A) + |\nu|(B)$

(picture) from linearity. (and choose $f = 1_A \pm 1_B$)

$\Rightarrow |\nu|(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty |\nu|(A_i)$

Step 3 $d\nu = g\, d\mu$ for some $g \in L^1(X,\mu)$

"baby Radon-Nikodym" $\Rightarrow d\nu = h\, d|\nu|$ $\quad |h| \le 1$

"the" Radon-Nikodym $\Rightarrow d|\nu| = \rho\, d\mu$ $\quad \rho \in L^1(X,\mu)$

$|\nu| \ll \mu$, $|\nu|(X) < \infty$, $|\nu(A)| \le |\nu|(A)$

$\Rightarrow g(x) = h(x)\rho(x)$ will do.

Step 4 $\varphi(f) = \varphi_g(f)$ for simple func f.

(cont.)

Step 5 $g \in L^q(X,\mu)$

$M < \infty$; put $A_M = \{x \in X : |g| \le M\}$

so $g_M = 1_{A_M} \cdot g \in L^\infty(X,\mu) \subset L^q(X,\mu)$

Then $\phi_{g_M} \in L^p(X,\mu)^*$ and

$$\|\phi_{g_M}\| = \sup_{\|f\|_p \le 1} |\phi_{g_M}(f)| = \sup_{\|f\|_p \le 1} |\varphi_g(1_{A_M} \cdot f)|$$

By density it's enough to take simple func. f.

$\Rightarrow 1_{A_M} \cdot f$ also simple, $\varphi_g(1_{A_M} \cdot f) = \psi(1_{A_M} \cdot f)$ Step 4.

So $\|\phi_{g_M}\| \le \sup_{f:\text{ simple},\ \|f\|_p \le 1} \|\psi\| \cdot \underbrace{\|1_{A_M} f\|_p}_{\le \|f\|_p} \le \|\psi\|$.

By "converse to Hölder" $\|g_M\|_q \le \|\psi\|$

$0 \le |g_M|^q \nearrow |g|^q$ by monotone conv. th'm

$$\|g\|_q = \lim_{M \to \infty} \|g_M\|_q \le \|\psi\|.$$

Step 6. $\psi = \varphi_g$. (density of simple funcs)

σ-finite case: reduction to finite case.

Take $X_1 \subset X_2 \subset \cdots$ meas., $X = \bigcup_{i=1}^\infty X_i$, $\mu(X_i) < \infty$

$L^p(X_i,\mu) \underset{\text{isomet.}}{\subset} L^p(X,\mu)$ by extension by 0 outside X_i

$\psi|_{L^p(X_i,\mu)} = \varphi_{g_i}$ for $g_i \in L^q(X_i,\mu)$

$\|g_i\|_q = \|\psi|_{L^p(X_i,\mu)^*}\| \le \|\psi\|$

$\Rightarrow g(x) = \lim_{i \to \infty} g_i(x)$ makes sense a.e.

$$|g(x)|^q = \lim_{i \to \infty} |g_i(x)|^q$$

$\underset{\text{Fatou's lem.}}{\Rightarrow} \int |g|^q d\mu \le \liminf \int |g_i|^q d\mu \le \|\psi\|^q$ ∎

Decomposition of signed measures

<u>Jordan</u> <u>decomposition</u> : ν signed measure.

$\Rightarrow$ $\nu = \nu_+ - \nu_-$ for (pos.) measures $\nu_+ \perp \nu_-$

How to get this (Thm 24.19)

Want to find $P \in \mathcal{M}$ (support of ν_+) s.t.

- $A \in \mathcal{M}$, $A \subset P \Rightarrow \nu(A) \geq 0$
- $A \in \mathcal{M}$, $A \subset N = P^c \Rightarrow \nu(A) \leq 0$

(Hahn <u>decomposition</u> Def 24.17). Then we can put

$\nu_+(A) = \nu(A \cap P)$, $\nu_-(A) = \nu(A \setminus P) = \nu(A \cap N)$

Strategy for finding P: (Lem 24.16, Thm 24.18)

Assume $\nu(A)$ doesn't take ∞ (otherwise look at $-\nu$)

Step 1. $\nu(E) > 0 \Rightarrow \exists A \subset E$: $\forall B \subset A$ $\nu(B) \geq 0$
$\nu(A) \geq \nu(E)$.

"A is positive"

Step 2 Put $s = \sup_{E \in \mathcal{M}} \nu(E)$. Then

$\exists$ pos. sets $P_1 \subset P_2 \subset \cdots$ $\nu(P_i) \to s$.

Step 3. $P = \bigcup_{i=1}^{\infty} P_i$ satisfies $\nu(P) = s$ (hence $s < \infty$)

Step 4 $N = P^c$ is "negative" $\forall B \subset N$ $\nu(B) \leq 0$

otherwise use Step 1 to get $P' \subset N$ pos.

$P \cup P'$ pos, $\nu(P \cup P') > s$. contra