( Hahn decomposition, Jordan decomposition from the note of last time )

Th'm ( Radon - Nikodym th'm for signed measures, 24.27 )

$(X, \mathcal{M}, \mu)$ $\sigma$-finite meas. sp.

$\nu$ : $\sigma$-finite signed measure

1. $\exists$ Lebesgue decomp. $\nu = \nu_a + \nu_s$ rel. to $\mu$.

2. $\exists$ $\rho$ meas. func. on $X$ s.t. $\rho = \rho^+ - \rho^-$

$\rho^{\pm} \geq 0$, $\rho^+ \in L^1(X, \mu)$ or $\rho^- \in L^1(X, \mu)$

$$\nu_a(A) = \int_A \rho \, d\mu = \underbrace{\int_A \rho^+ d\mu - \int_A \rho^- d\mu}_{\text{at most one of these is } \infty}$$

( 3. $\nu_s = 0 \iff \nu \ll \mu$ ( i.e. $\mu(A) = 0$, $B \subset A \Rightarrow \nu(B) = 0$ ) )

Outline. Take Jordan decomp $\nu = \nu_+ - \nu_-$.

Usual Radon - Nikodym ( 24.13 on Sept. 21 ) :

$\nu_{\pm} = \nu_{\pm, a} + \nu_{\pm, s}$ Lebesgue decomp

$d\nu_{\pm} = \rho^{\pm} d\mu$ for some $\rho^{\pm} : X \to [0, \infty)$

Then $\nu_a = \nu_{+,a} - \nu_{-,a}$, $\nu_s = \nu_{+,s} - \nu_{-,s}$

$\rho = \rho^+ - \rho^-$ will do.

Step I. $\nu_a$, $\nu_s$ as above are signed measures.

To check : The range of $\nu_a$ does not contain both $\infty$ and $-\infty$.

Same for $\nu_s$.

$\Leftarrow$) Either : the range of $\nu_+$ does not cont. $\infty$

or ~~~ $\nu_-$ ~~~ $\infty$.

Step 2 $\nu_a \ll \mu$, $\nu_s \perp \mu$. ($\Rightarrow$ they give Leb. dec. of $\nu$)

From corresp. properties of $\nu_{\pm,a}$, $\nu_{\pm,s}$

Step 3. $p^+ \in L^1(X,\mu)$ or $p^- \in L^1(X,\mu)$

From $\nu_+(X) < \infty$ or $\nu_-(X) < \infty$ ∎

Density, again (§22.1).

Recall : Stone-Weierstrass th'm and quotient of compact spaces.

$X$ : compact Hausdorff space (lik $[0,1]^k$)

$C_{\mathbb{R}}(X) = \{ f : X \to \mathbb{R} \text{ cont.} \}$, norm $\|f\| = \max_{x \in X} |f(x)|$

$A \subset C(X)$ is dense if :

1. $A$ is a subalg. with scalars ($1 \in A$, $f_1, f_2 \in A \Rightarrow f_1 + f_2, f_1 \cdot f_2$, $r \cdot f_1 \in A$)

2. $\forall x \neq x' \in X$ $\exists f \in A$ $f(x) \neq f(x')$.
   ($A$ separates points of $X$)

What if we do not assume 2. ?

$\rightsquigarrow$ equiv. rel. $x \underset{A}{\sim} x' \equiv \forall f \in A$ $f(x) = f(x')$

$\Rightarrow$ graph of $\sim_A$ is closed.

$\Rightarrow Y = X/\sim_A$ is (compact) Hausdorff

$p : X \to Y$ defs. $p^* : C_{\mathbb{R}}(Y) \hookrightarrow C_{\mathbb{R}}(X)$.

$A$ is a dense subset of $p^*(C_{\mathbb{R}}(Y))$

Ex. $X = [0,1] \times [0,1]$, $A = \{ f(x,y) = p(x)$ for some polynom. $p \}$.

$(x,y) \sim_A (x',y') \Leftrightarrow y = y'$. $Y \simeq [0,1]$.

Analogue of this for measurable spaces.

Want to understand : sub $\sigma$-algs. one gets :

- by "dropping some information" from a probability space.
- by taking subalgs of measurable funcs. ...

Thm (22.4) $(X, \mathcal{M}, \mu)$ measure space

$1 \le p < \infty$, $A \subset L^p(X, \mathcal{M}, \mu)$ s.t.

1. $f_1, f_2 \in A$, $r \in \mathbb{R} \Rightarrow rf_1 + f_2, f_1 f_2 \in A$.
   (A is an algebra)
2. $\exists$ sequence $\varphi_1, \varphi_2, \cdots \in A$ s.t. $\exists C > 0$
   $|\varphi_k(x)| \le C$, $\varphi_k(x) \to 1$ a.e.
   "approximate unit"
3. the <u>smallest</u> $\sigma$-alg $\mathcal{M}'$ on $X$ s.t.
   $\forall f \in A$ is $\mathcal{M}'$-measurable
   is equal to $\mathcal{M}$ (always $\mathcal{M}' \subset \mathcal{M}$).
   " A separates points (or measurable sets)"

Then A is dense in $L^p(X, \mathcal{M}, \mu)$

(Rem: without assumption $\mathcal{M}' = \mathcal{M}$ : A is dense in $L^p(X, \mathcal{M}', \mu)$.)