Exercise set 2

Problem 3 We want to write the inverse of $\ell^\infty \to (\ell^1)^*$.

$\delta_n = (0, 0, \cdots, \underset{n\text{-th}}{1}, 0, \cdots)$

(1) Claim : $|\phi(\delta_n)| \leq \|\phi\|$.

Generally $|\phi(x)| \leq \|\phi\| \cdot \|x\|_1$ for $x \in \ell^1(\mathbb{N})$

$\|\delta_n\|_1 = \sum_{i=1}^\infty |\delta_n(i)| = 1$.

(2). Put $a_n = \phi(\delta_n)$. Then $(a_n)_{n=1}^\infty \in \ell^\infty(\mathbb{N})$

by (1). μ : counting measure on $\mathbb{N}$

$g \in L^\infty(\mathbb{N}, \mu) \simeq \ell^\infty(\mathbb{N})$ corresp. to $(a_n)_n$

(so $g(n) = a_n = \phi(\delta_n)$)

Claim : $\phi_g = \phi$, so $\phi \rightsquigarrow (a_n)_n$ is the inverse of $g \rightsquigarrow \phi_g$ up to the above corresp. between $(a_n)_n$ and g.

Proof of claim :

- ϕ_g and ϕ agrees on linear span of δ_n $(n = 1, 2, \cdots)$

$\because$ $\phi_g(\delta_n) = \sum_{i=1}^\infty g(i)\delta_n(i) = g(n) = \phi(\delta_n)$

- linear span of δ_n $(n=1,2,\cdots)$ is dense in $\ell^1(\mathbb{N})$.

$\because$ $(b_n)_n \in \ell^1(\mathbb{N}) \Rightarrow \underbrace{\sum_{n=1}^\infty b_n \delta_n}_{i\text{-th comp. is } b_i} = (b_i)_{i=1}^\infty$

$\sum_{n=1}^N b_n \delta_n \to \sum_{n=1}^\infty b_n \delta_n$. in $\ell^1\mathbb{N}$.

(3). $\|\delta_n\|_p = \left(\sum_{i=1}^\infty |\delta_n(i)|^p\right)^{1/p} = 1$.

So $\phi \in \ell^p(\mathbb{N})^* \Rightarrow a_n = \phi(\delta_n)$ is bounded.

$g(n) = a_n \rightsquigarrow g$ (bdd) measurable func. on $\mathbb{N}$

converse to Hölder's inequality implies that

but $\phi_g = \phi$ on finitely supp. funcs $\to \|g\|_q < \infty$

if $\|g\|_q = \infty$ then $\exists f_1, f_2, \cdots$ $\|f_i\|_p \le 1$, finite support, $|\phi_g(f_i)| \to \infty$

So $g \in \ell^q(\mathbb{N})$, $\phi \rightsquigarrow g$ is the inverse of

$\ell^q(\mathbb{N}) \to \ell^p(\mathbb{N})^*$, $g \rightsquigarrow \phi_g$.

Problem 4.

We will write $\int f(t)\, d\mu(t) = \int_0^1 f(t)\, dt$.

(1) We need to find $C > 0$ s.t. $\|T_K f\|_{L^1} \le C \|f\|_{L^1}$

Claim $C = \max_{0 \le s,t \le 1} K(s,t)$ will do.

$$\|T_K f\|_{L^1} = \int_0^1 |(T_K f)(s)|\, ds = \int_0^1 \left| \int_0^1 K(s,t) f(t)\, dt \right| ds \cdots (*)$$

Minkowski's inequality for integrals implies

$$\left| \int_0^1 K(s,t) f(t)\, dt \right| \le \int_0^1 |K(s,t) f(t)|\, dt$$

so $(*) \le \int ds \int dt\, |K(s,t) f(t)| \cdots (*)'$

Tonelli's th'm says $(*)' = \int_0^1 dt \int_0^1 ds\, |K(s,t) f(t)|$

$$\int |K(s,t) f(t)|\, ds \le C |f(t)|$$

So $(*)' \le C \int_0^1 f(t)\, dt = C \|f\|_{L^1}$

(2) The same C will do, but ~~we~~ need $\|T_K f\|_p \le \|f\|_p$

Take q s.t. $\frac{1}{p} + \frac{1}{q} = 1$

so $|K(s,t) f(t)| = |K(s,t)|^{\frac{1}{q}} \left(|K(s,t)|\, |f(t)|^p \right)^{\frac{1}{p}}$

$$\int_0^1 |K(s,t) f(t)|\, dt \underset{\text{Hölder}}{\le} \left(\int_0^1 |K(s,t)|\, dt \right)^{\frac{1}{q}} \left(\int_0^1 |K(s,t)|\, |f(t)|^p\, dt \right)^{\frac{1}{p}}$$

$$\|T_K f\|_p^p = \int |(T_K f)(s)|^p \, ds$$

$$= \int \left| \int K(s,t) f(t) \, dt \right|^p ds$$

$$\leq \int \left(\int |K(s,t) f(t)| \, dt \right)^p ds$$

$$\leq \int \Big(\big(\underbrace{\int_0^1 |K(s,t)| \, dt}_{\leq C} \big)^{1/q} \big(\underbrace{\int_0^1 |K(s,t)| \, |f(t)|^p \, dt}_{\leq C \int_0^1 |f(t)|^p dt} \big)^{1/p} \Big)^p ds$$

$$\leq \int C^p \left(\int_0^1 |f(t)|^p \, dt \right)^p ds = C^p \|f\|_p^p .$$

Take p-th root to get $\|T_K f\|_p \leq C \|f\|_p$.