Conditional expectation

$(X, \mathcal{M}, \mu)$: σ-finite measure space

$\mathcal{M}' \subset \mathcal{M}$ sub σ-algebra. (s.t. $(X, \mathcal{M}', \mu)$ still σ-fin.)

$\leadsto$ $L^p(X, \mathcal{M}', \mu) \subset L^p(X, \mathcal{M}, \mu)$ Banach subsp.

Formal requirement (characterization) of conditional expectation

linear map $E : L^p(X, \mathcal{M}, \mu) \to L^p(X, \mathcal{M}', \mu)$

- $\|E\| \leq 1$ i.e. $\|E(f)\|_p \leq \|f\|_p$
- $E(f) = f$ for $f \in L^p(X, \mathcal{M}', \mu)$

Motivation : μ probability measure

$\leadsto$ so we are looking at a probability space

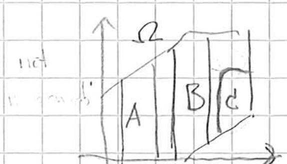
$(\Omega, \mathcal{M}, \mathbb{P})$

Ω ↑ set of samples; $\mathbb{P}$ ← encodes probability

$X, Y, \ldots$ random variables (measurable funcs. on Ω)
(not the "first X" as space)

$\mathcal{M}$: encoding what can be "measured" with all rand. vars.

$\leadsto$ $\mathcal{M}'$: encoding "limited information" with restricted number of rand. vars.



$A, B \in \mathcal{M}'$ but not $C, \ldots$

detectable direction. (Ω' below)

$E[X \mid \mathcal{M}'], E[Y \mid \mathcal{M}'], \ldots$: $\mathcal{M}'$-measurable approximation of $X, Y, \ldots$

Rough idea : $\Omega' = $ "$\Omega / \mathcal{M}'$" $\quad \Omega \xrightarrow{\pi} \Omega'$

$$E[X \mid \mathcal{M}'](\omega') = \text{"} \int_{\{\omega : \pi(\omega) = \omega'\}} X(\omega) \, d\mathbb{P}(\omega) \text{"}$$

← OK if this has pos. meas.

Formally : (back to σ-finite $(X, \mathcal{M}', \mu)$, $\mathcal{M}' \subset \mathcal{M}$.)

construction of $E : L^q(X, \mathcal{M}, \mu) \to L^p(X, \mathcal{M}', \mu)$

for $1 < q \leq \infty$. ($\frac{1}{p} + \frac{1}{q} = 1$ for $1 \leq p < \infty$)

Recall : $L^q(X, \mathcal{M}, \mu) \cong L^p(X, \mathcal{M}, \mu)^*$

$$g \longleftrightarrow (\phi_g : \underset{\underset{L^p(X, \mathcal{M}, \mu)}{\cap}}{f} \mapsto \int f \cdot g \, d\mu)$$

Same for $\mathcal{M}'$.

Given $g \in L^q(X, \mathcal{M}, \mu)$ take $\psi = \phi_g|_{L^p(X, \mathcal{M}', \mu)}$

Claim ψ is a bounded linear functional

on $L^p(X, \mathcal{M}', \mu)$, $\|\psi\| \leq \|\phi_g\| = \|g\|_q$

- linearity is obvious from def.
- boundedness : $\|\psi\| = \sup_{\substack{f \in L^p(X, \mathcal{M}', \mu) \\ \|f\|_p \leq 1}} |\psi(f)| \leq \sup_{\substack{f \in L^p(X, \mathcal{M}, \mu) \\ \|f\|_p \leq 1}} |\phi_g(f)| = \|\phi_g\|$.

$\rightsquigarrow$ $\psi = \phi_{E(g)}$ for some $E(g) \in L^q(X, \mathcal{M}', \mu)$

Then $g \mapsto E(g)$ is linear, contractive
$\|E\| \leq 1$.

- linearity $E(rg_1 + g_2)$ represents

$$\phi_{rg_1+g_2}|_{L^p(X, \mathcal{M}', \mu)} = (r\phi_{g_1} + \phi_{g_2})|_{L^p(X, \mathcal{M}', \mu)}$$

$$= r\phi_{E(g_1)} + \phi_{E(g_2)}$$

i.e. $\phi_{E(rg_1+g_2)} = \phi_{rE(g_1)+E(g_2)}$

$\Rightarrow E(rg_1 + g_2) = rE(g_1) + E(g_2)$

- norm estimate $\|E(g)\|_q = \|\psi\| \leq \|\phi_g\| = \|g\|_q$