Pushforward measure

$(X, \mathcal{M}) \xrightarrow{f} (Y, \mathcal{N})$: map of measurable spaces

μ : measure on X

Def. The <u>pushforward</u> of μ by f (write $f_*\mu$)

is the measure on Y given by

$(f_*\mu)(A) = \mu(\underbrace{f^{-1}(A)}_{\text{in } \mathcal{M} \text{ by measurability of } f})$. $(A \in \mathcal{N})$

Ex. $(\Omega, \mathcal{M}, \mathbb{P})$ probability space

X random variable (measurable map $\Omega \to \mathbb{R}$)

$\rightsquigarrow \underbrace{\mu_X}_{\text{measure on } \mathbb{R}} = X_* \mathbb{P}$: distribution of X.

Hahn-Banach theorem (§25.1)

Want : X : Banach space $\Rightarrow X^*$ big enough.

more precisely $Y \subset X$ subspace, $\phi \in Y^*$

$\Rightarrow \exists \tilde{\phi} \in X^*$ s.t. $\tilde{\phi}|_Y = \phi$

When is this obvious ?

$(A, \mathcal{M}, \mu)$ σ-finite measure space

$\mathcal{M}' \subset \mathcal{M}$ σ-subalg, $(A, \mathcal{M}', \mu)$ still σ-fin.

$Y = L^p(A, \mathcal{M}', \mu) \subset L^p(A, \mathcal{M}, \mu) = X$

$1 \le p < \infty$. $\forall \phi \in Y^*$ is of the form ϕ_g

for $g \in L^q(A, \mathcal{M}', \mu)$ $\rightsquigarrow$ we can take $\tilde{\phi} = \phi_g$.

$(\subset L^q(A, \mathcal{M}, \mu))$

Not so obvious : $X = \ell^\infty(\mathbb{N}) = \{(a_i)_{i=1}^\infty$ bounded seq$\}$

$Y = \{(a_i)_{i=1}^\infty : \lim_{i\to\infty} a_i$ exists $\}$

$\phi \in Y^*$ by $\phi((a_i)_i) = \lim a_i$. $|\phi((a_i)_i)| \le \sup|a_i|$

$\|\phi\| = 1$ by

(cont.) $\tilde{\phi} \in X^*$ would be a generalized limit scheme for all bounded sequences (Banach lim.)

Conceptually: $Z = \ell^\infty(\mathbb{N}) / c_0(\mathbb{N})$ (will be Banach space). $\rightsquigarrow$ $\phi \in Z^*$ $\phi([(a_i)_i])$ only depends on "asymptotic behavior" of $(a_i)_i$

i.e. if $a_i - b_i \to 0$ $(i \to \infty)$ then

$\phi([(a_i)_i]) = \phi([(b_i)_i])$.

Def (25.2) X: vector space over $\mathbb{R}$ (or $\mathbb{C}$)

a Minkowski functional is a map $p: X \to \mathbb{R}$ s.t.

- $p(x+y) \le p(x) + p(y)$
- $p(cx) = c\,p(x)$ if $c \ge 0$.

Ex. seminorms, linear functionals (over $\mathbb{R}$), ...

Th'm (Hahn-Banach, 25.4)

Given X: real vector space, p: Minkowski func. on X

$M \subset X$ subsp. $\phi: M \to \mathbb{R}$ linear, $\phi(x) \le p(x)$.

Then $\exists\, \tilde{\phi}: X \to \mathbb{R}$ linear, $\tilde{\phi}(x) = \phi(x)$ for $x \in M$,

$\tilde{\phi}(x) \le p(x)$ for $x \in X$.

How to use this: $\underset{X \text{ Ban. sp.}}{F}$ $\phi \in M^*$, put $p(x) = \|\phi\| \cdot \|x\|$.

$\Rightarrow$ get $\tilde{\phi} \in X^*$ as F., $\|\tilde{\phi}\| = \|\phi\|$.

M could be finite dimensional subsp. of X.

$\rightsquigarrow$ we have $\tilde{\phi} \in X^*$ with arbitrary prescribed behavior on M.

Idea: 1. ϕ extends to $M + \mathbb{R}x$ (1-dim bigger)

2. "Keep repeating" this to reach X.

Proof Step 1. $x \in X \setminus M \Rightarrow \exists\ \phi' : M + \mathbb{R}x \xrightarrow{\text{lin.}} \mathbb{R}$

s.t. $\phi'(y+\lambda x) \le p(y+\lambda x)$, $\phi'(y) = \phi(y)$ $\quad y \in M$. $\lambda \in \mathbb{R}$.

Work out the condition for $\alpha = \phi'(x)$:

$$\phi(y) + \lambda\alpha = \phi'(y+\lambda x) \le p(y+\lambda x).$$

First reduction : (Divide by $|\lambda|$)

$$\phi(y) + \varepsilon\alpha \le p(y+\varepsilon x) \quad \text{for } \varepsilon = \pm 1$$

i.e. $\underset{(\varepsilon=1)}{\alpha \le p(y+x) - \phi(y)}$, $\quad \underset{(\varepsilon=-1)}{\phi(y') - p(y'-x) \le \alpha}$

This has sol. iff $\sup_{y' \in M} \phi(y') - p(y'-x) \le \inf_{y \in M} p(y+x) - \phi(y)$

$\Leftrightarrow \phi(y+y') \le p(y+x) + p(y'-x) \quad \forall y, y' \in M$

$\phi \le p$ on M gives $\phi(y+y') \le p(\underbrace{y+y'}_{y+x+y'-x})$ ok.

Step 2. existence of $\tilde{\phi}$ on X.

Morally : repeat Step 1 "infinitely many times"

Formally : get "maximal extension" of ϕ.
(domain is as big as possible)
& use Step 1 to say that its dom. is X.

Take all possible extensions of ϕ :

$$F = \{ (N, \phi') : M \subset N \subset X \text{ subsp} \\ \phi' : N \to \mathbb{R} \text{ lin. } \phi'(y) = \phi(y) \quad y \in M. \\ \phi' \le p \text{ on } N \}$$

Order rel on F : $(N_1, \phi_1') \le (N_2, \phi_2')$

iff $N_1 \subset N_2$, $\phi_2'|_{N_1} = \phi_1'$.

We can use Zorn's lemma to get a maximal elem. $(N, \phi') \in F$.

Claim $N = X$ (so $\tilde{\phi} = \phi'$ will do)

Otherwise take $x \in X \setminus N$ and use step 1 to get ϕ'' on $N + \mathbb{R}x$.

$\Rightarrow (N + \mathbb{R}x, \phi'') \underset{\neq}{\geq} (N, \phi')$, contradicts maximality. ☒

Zorn's lem: $(P, \leq)$ partially ordered set s.t. $\forall A \subset P$ totally ordered (chain)

$\exists u \in P \ \forall a \in A \quad a \leq u$ (upper bound)

Then $\exists m \in P$ s.t. $m' \geq m \Rightarrow m' = m$ (maximal)

For $P = \mathcal{F}$, given $A \subset \mathcal{F}$ we get u by

$$N_u = \bigcup_{(N,\phi') \in A} N, \quad \phi_u(y) = \phi'(y) \text{ if } y \in N.$$