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Applications of the Hahn-Banach th'm.

Complex version

Prop. X : Banach space, $x \in X$.

Then $\exists \phi \in X^*$ s.t. $\|\phi\| = 1$, $\phi(x) = \|x\|$.

Proof Take $M = \mathbb{R}x$, $p(y) = \|y\|$ $y \in X$.

$\phi_0(\lambda x) = \lambda \|x\|$ on M. $\Rightarrow$ $\phi_0 \leq p$.

Hahn-Banach $\Rightarrow$ $\exists \phi \in X^*$ $\phi|_M = \phi_0$, $\phi \leq p$.

i.e. $\phi(x) = \|x\|$, $|\phi(y)| \leq \|y\|$. $\boxed{}$

This can be used to reduce claims about vector valued functions to scalar ones.

Example : Liouville's theorem.

bounded entire functions are constant.

Holomorphic functions : $f(z) \in X$

X : (complex) Banach sp., z : complex variable.

f is holom. at z_0 if

$$(f'(z_0) =) \lim_{w \to 0} \frac{f(z_0+w) - f(z_0)}{w} \text{ exists.}$$

Prop. $f : \mathbb{C} \to X$ holomorphic, bounded

$(\exists C > 0 \ \forall z \in \mathbb{C} \ \|f(z)\| \leq C)$

then f is constant.

Proof. For any $\varphi \in X^*$, $\varphi \circ f : \mathbb{C} \to \mathbb{C}$ is holom. in the usual sense, bdd by $\|\varphi\| \cdot C$.

$\underset{\text{Liouville}}{\Rightarrow}$ $\varphi \circ f$ is const.

If $f(z) \neq f(w)$, $\exists \varphi$ s.t. $\varphi(f(z) - f(w)) \neq 0$ $\boxed{}$

Hahn-Banach for complex Banach spaces

Theorem (Hahn-Banach over $\mathbb{C}$, 25.5)

X : complex vector space

p : seminorm on X. ($p: X \to [0,\infty)$, $p(\lambda x) = |\lambda| p(x)$
$p(x+y) \le p(x) + p(y)$)

$M \subset X$ subspace, $\varphi : M \to \mathbb{C}$ lin, $|\varphi(x)| \le p(x)$

Then $\exists\ \tilde{\varphi} : X \to \mathbb{C}$ lin. $|\tilde{\varphi}(x)| \le p(x)$.

Key obs.: $X_{\mathbb{R}} = X$ but as a real vec. sp.

$\xi : X_{\mathbb{R}} \to \mathbb{R}$ lin $\rightsquigarrow$ $\psi(x) = \xi(x) - i\,\xi(ix)$
is linear as $X \to \mathbb{C}$.

$\xi = \operatorname{Re} \psi : X_{\mathbb{R}} \to \mathbb{R} \longleftrightarrow \psi : X \to \mathbb{C}$ lin

Proof.

From φ on M: $\xi(x) = \operatorname{Re} \varphi(x)$ satisfies

$|\xi(x)| \le |\varphi(x)| \le p(x)$.

$\underset{\text{HB for } X_{\mathbb{R}}}{\Rightarrow}$ $\exists\ \tilde{\xi} : X_{\mathbb{R}} \to \mathbb{R}$ lin, $|\tilde{\xi}(x)| \le p(x)$

Put $\tilde{\varphi}(x) = \tilde{\xi}(x) - i\,\tilde{\xi}(ix)$.

Claim $|\tilde{\varphi}(x)| \le p(x)$.

By $\mathbb{C}$-linearity $|\lambda| = 1 \Rightarrow |\tilde{\varphi}(\lambda x)| = |\tilde{\varphi}(x)|$

Choose $|\lambda| = 1$ s.t. $\tilde{\varphi}(\lambda x) \in \mathbb{R}$

this means $\tilde{\varphi}(\lambda x) = \xi(\lambda x) \underset{\text{abs.}}{\le} p(\lambda x) = p(x)$

Prop. $X \hookrightarrow X^{**}$ isometrically if X is normed vec. sp.

Rem. X^{**} is always Banach.

$\rightsquigarrow$ closure of X in X^{**} is a model of completion.

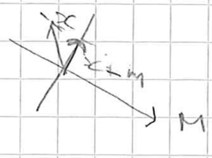
Quotient space, adjoint map (§25.1.2)

X : Banach sp.

$M \subset X$ closed subspace

Def The (semi) norm on X/M :

$$\|[x]\| = \inf_{m \in M} \|x - m\|.$$



Thm (25.17) X/M is a Banach sp. (with the above norm)

Proof Step 1 $\|[x]\| \neq 0$ if $[x] \neq 0$. ($x \notin M$)

$\|[x]\| = 0$ means $\inf_{m \in M} \|x - m\| = 0$ i.e.

$\exists$ seq. $m_1, m_2, \ldots \in M$ s.t. $\|x - m_i\| \to 0$.

M is closed $\Rightarrow$ $x = \lim_{i \to \infty} m_i \in M$.

Step 2. X/M is complete for $\|[x]\|$

Enough to show: $\sum_{i=1}^{\infty} \|[x_i]\| < \infty \Rightarrow \exists \sum_{i=1}^{\infty} [x_i] \in X/M$.

Choose $m_i \in M$ s.t. $\|[x_i]\| \geq \frac{1}{2} \|x_i - m_i\|$.

$\Rightarrow \sum_{i=1}^{\infty} \|x_i - m_i\| \leq 2 \sum \|[x_i]\| < \infty$.

So $y = \sum_{i=1}^{\infty} (x_i - m_i)$ exists in X.

Then $[y] = \sum_{i=1}^{\infty} [x_i]$. Indeed:

$$\|[y] - \sum_{i=1}^{N} [x_i]\| = \|[\sum_{i=N+1}^{\infty} x_i]\| \leq \|\sum_{i=N+1}^{\infty} x_i\| \xrightarrow[N \to \infty]{} 0 \quad \blacksquare$$

Prop. (25.17) $T : X \to Y$ vanishes on M

$\Leftrightarrow$ $T = T' \circ \pi$ for $T' : X/M \to Y$.

where $\pi : X \to X/M$ is $x \mapsto [x]$.

Def (25.17) X, Y : Banach sp.

$T : X \to Y$ bounded lin. op

The <u>adjoint</u> (or <u>transpose</u>) of T is

T^* (or T^t) $: Y^* \to X^*$, $\phi \mapsto \phi \circ T$.

Prop (part of 25.16) $\|T^*\| = \|T\|$

Proof $\|T^*\| = \sup_{\phi \in Y^*, \|\phi\|=1} \|T^*\phi\| = \sup_{\substack{\phi \in Y^*, \|\phi\|=1 \\ x \in X, \|x\|=1}} |\phi(Tx)|$

$\underset{HB}{=} \sup_{x \text{ as above}} \|Tx\| = \|T\|.$

Rem. For Hilbert spaces, we had another

conv. $T : H_1 \to H_2 \rightsquigarrow T^* : H_2 \to H_1$ characterized

by $(Tx, y)_{H_2} = (x, T^*y)_{H_1}$

real Hilb sp. : agrees with above up to

identification $H \simeq H^*$ $x \leftrightarrow (\phi_x : x' \mapsto (x, x'))$

complex Hilb. sp : $\phi_y : y' \mapsto (y', y)_H$

$T^* \phi_y = \phi_{T^*y}$

for functional   for orig. Hilb. sp.