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Open Mapping Theorem (

X, Y : Banach spaces, $T: X \to Y$ surjective bounded linear map. $\Rightarrow$ T is an open map.
i.e. $V \subset X$ open $\Rightarrow$ $T(V) \subset Y$ open.

What are open maps? $\mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x$.

$C([0,1]) \to \mathbb{R}$, $f \mapsto f(0)$
norm $\|f\|_\infty$.
or $f \mapsto \int_0^1 f(t)\,dt$.

conditional expectation $L^p(X, \mathcal{M}, \mu) \to L^p(X, \mathcal{M}', \mu)$
for $\mathcal{M}' \subset \mathcal{M}$ σ-fin. for μ.

What are not open? $C_c(\mathbb{N}) \xrightarrow{\text{incl}} c_0(\mathbb{N}), \ell^p(\mathbb{N}), \dots$
fin. supp. seqs (img is not open)

H ∞-dim Hilb. sp. $\underset{\text{norm top}}{H} \xrightarrow{\text{id}} \underset{\text{"weak top"}}{H}$

How do we want to use this?

H_1, H_2 Hilb. sp. $T \in L(H_1, H_2)$ bdd lin op. TFAE.

1. $\ker(T)$, $\ker(T^*)$ fin dim, $T(H_1) \subset H_2$ closed.
   (T is a Fredholm operator)
2. $\exists\, S \in L(H_2, H_1)$ s.t. $Id_{H_1} - TS$, $Id_{H_2} - ST$
   are (in the norm closure of) finite rank ops.

Key : $T|_{\ker(T^*)^\perp} : \ker(T^*)^\perp \to T(H_1)$ is invertible
by open mapping thm.

Example $D(f) = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2} + \sum a_i \frac{\partial f}{\partial x_i}$, $W^{k+2,2}(\mathbb{R}^n) \to W^{k,2}(\mathbb{R}^n)$
$\mathbb{T}^n$ $\mathbb{T}^n$
is Fredholm.

Key ingredient : Baire's category theorem (16.2)

X complete metric space.
$V_1, V_2, \dots$ open dense sets in X.
Then $G = \bigcap_{i=1}^\infty V_i$ is still dense in X (G_δ-dense set)

Proof : want : $W \neq \emptyset$ open in $X \Rightarrow W \cap G \neq \emptyset$.

We'll construct $\underbrace{\text{a seq. } \varepsilon_1 > \varepsilon_2 > \cdots \to 0,}$ a seq. $x_1, x_2, \cdots$ in W s.t.

- $d(x_i, x_k) \leq \varepsilon_i$ for $k \geq i$ ($(x_i)_i$ Cauchy)
- $x_\infty = \lim_{i \to \infty} x_i$ (exists in X) is in $W \cap G$.

Step 1. construction of $(x_i)_{i=1}^{\infty}$ & $(\varepsilon_i)_{i=1}^{\infty}$

1-1 x_1 & ε_1 : V_1 dense $\Rightarrow \exists\, x_1 \in W \cap V_1$

$W \cap V_1$ open $\Rightarrow \exists\, \varepsilon_1$ s.t. $\underbrace{\overline{B(x_1, \varepsilon_1)}}_{\{x' : d(x_1, x') < \varepsilon_1\}^{\text{clos.}}} \subset W \cap V_1$

1-2 x_2 & ε_2 : V_2 dense $\Rightarrow \exists\, x_2 \in B(x_1, \varepsilon_1) \cap V_2$

$B(x_1, \varepsilon_1) \cap V_1$ open $\Rightarrow \exists\, \varepsilon_2 (< \frac{1}{2}\varepsilon_1)$ s.t. $\overline{B(x_2, \varepsilon_2)} \subset B(x_1, \varepsilon_1) \cap V_2$.

Note $x_2 \in W \cap V_2$

... keep repeating, so $\varepsilon_{k+1} < \frac{1}{2}\varepsilon_k$, $\overline{B(x_{k+1}, \varepsilon_{k+1})} \subset B(x_k, \varepsilon_k) \cap V_{k+1} \subset W \cap V_{k+1}$

Step 2 $x_\infty \in W \cap G$

We have $\overline{B(x_1, \varepsilon_1)} \supset \overline{B(x_2, \varepsilon_2)} \supset \cdots \searrow \{x_\infty\}$.

So $x_\infty \in \overline{B(x_k, \varepsilon_k)} \subset W \cap V_k$ for all k.

Proof of Open Mapping Thm.

Put $B^X_\alpha = B^X(0_X, \alpha) = \{ x \in X : \| x \| < \alpha \}$, etc

and $E_\alpha = T(B^X_\alpha) \subset Y$.

Step 1 $\exists\, y \in Y, \varepsilon > 0, n \in \mathbb{N}$ s.t. $\overline{B^Y(y, \varepsilon)} \subset \overline{E_n}$

Otherwise $\overline{E_n}^c = Y \setminus \overline{E_n}$ (open) is dense for all n.

$\underset{\text{Baire}}{\Longrightarrow}$ $\bigcap_{n=1}^{\infty} \overline{E_n}^c$ is still dense. but this

contradicts with $\bigcup_{n=1}^{\infty} E_n = T(X) = Y$.

Step 2 With ε, n from Step 1 $B^Y_\varepsilon \subset \overline{E_{2n}}$.

Take y also from Step 1

$y' \in B^Y_\varepsilon \Rightarrow y + y' \in B^Y(y, \varepsilon) \subset \overline{E_n}$

Claim $\forall \varepsilon' \ \exists x_1, x_2 \in B^X_n$ s.t. $\| T(x_1 - x_2) - y' \| < \varepsilon'$

$(\Rightarrow y' \in \overline{E_{2n}}$ i.e. $B^Y_\varepsilon \subset \overline{E_{2n}})$

Proof of claim $\underset{y,\ y+y'}{B^Y(y, \varepsilon)} \subset \overline{E_n}$ means

$\exists x_1, x_2 \quad \| T x_1 - (y + y') \|, \underbrace{\| T x_2 - y \|}_{\| - T x_2 + y \|} < \frac{1}{2} \varepsilon'$

$\| T x_1 - T x_2 - (y + y') + y \|$

$\leq \| T x_1 - y + y' \| + \| - T x_2 + y \| < \varepsilon'$.

Step 3 $\delta = \frac{\varepsilon}{2n} \rightsquigarrow B^Y_{\delta\alpha} \subset \overline{E_\alpha}$

from linearity of T

Step 4 $B^Y_{\delta\alpha} \subset E_\alpha$. with above δ.

Again by linearity enough to establish $B^Y_\delta \subset E_1$

Fix $y \in B^Y_\delta$ i.e. $\|y\| < \delta$.

Choose $\alpha_1, \alpha_2, \cdots > 0$ s.t. $\|y\| < \delta\alpha_1 < \delta$, $\sum_{i=1}^\infty \alpha_i < 1$

and then $x_1, x_2, \cdots \in X$ s.t.

$x_1 \in B^X_{\alpha_1} \quad \|y - T x_1\| < \alpha_2 \delta \quad (y \in B^Y_{\delta\alpha_1} \subset \overline{E_{\alpha_1}})$

$x_k \in B^X_{\alpha_k}$, $\cdots$ s.t. $\| y - \sum_{k=1}^n T x_k \| < \alpha_{n+1} \delta$

$\sum \alpha_i < 1 \Rightarrow x' = \sum_{i=1}^\infty x_i$ exists in X, $\|x'\| < 1$.

$y = T x'$ from above estim. i.e. $y \in E_1$.

Step 5 T is open, i.e. $V \subset X$ open $\Rightarrow TV \subset Y$ open.

Fix $x \in V$, take α s.t. $V \supset B^X(x, \alpha) = x + B^X_\alpha$.

then $TV \supset Tx + E_\alpha$ by linearity.

(cont.) $Tx + E_\alpha \supset Tx + B^Y_{\delta\alpha}$ by Step 4.

i.e. TV contains a neigh. of Tx.

this means TV is open. $\boxtimes$