Exercise set 3

Problem 5 :

(1) $\| f(b) - f(a) \| = \max\limits_{\phi \in X^*, \|\phi\| \le 1} | \phi(f(b)) - \phi(f(a)) |$

Combine

$\cdot$ $\|x\| = \max\limits_{\phi \in X^*, \|\phi\| \le 1} |\phi(x)|$ for $x \in X$

from Hahn-Banach

for $x = f(b) - f(a)$

$\cdot$ $\phi(f(b) - f(a)) = \phi(f(b)) - \phi(f(a))$

from linearity of ϕ

(2) for scalar valued C^1-class func. $g(t)$

$$|g(b) - g(a)| \le \int_a^b |g'(t)| dt$$

from $\left| \int_a^b g'(t)\, dt \right| \le \int_a^b |g'(t)| dt$.

with $g(t) = \phi(f(t))$ $\quad (\phi \in X^*, \|\phi\| \le 1)$

$$g'(t) = \lim_{h \to 0} \frac{g(t+h) - g(t)}{h} \underset{\text{linearity of } \phi}{=} \lim_{h \to 0} \phi\left(\frac{f(t+h) - f(t)}{h} \right) \underset{\text{continuity of } \phi}{=} \phi(f'(t))$$

so $|\phi(f(b)) - \phi(f(a))| \le \int_a^b |\phi(f'(t))| dt$.

by $\|\phi\| \le 1$ we have $|\phi(f'(t))| \le \|f'(t)\|$

Problem 6

(1) fix $0 \ne x_1 \in M$, define $\phi : M \to \mathbb{R}$ by

$\phi(\lambda x_1) = \lambda$

Hahn-Banach th'm gives $\tilde{\phi} \in X^*$ extending ϕ.

Set $N = \ker \tilde{\phi}$; closed by continuity of $\tilde{\phi}$

$X = M \oplus N$. indeed $x \in X$ can be written

as $\underbrace{\tilde{\phi}(x) x_1}_{\text{in } M} + \underbrace{(x - \tilde{\phi}(x) x_1)}_{\text{in } N}$, $M \cap N = 0$.

(2) take a basis $(x_i)_{i=1}^n$ in M. and

$\phi_i : M \to \mathbb{R}$ defined by $\phi_i(x_j) = \delta_{ij}$

take extensions $\tilde{\phi}_i$ to X.

Set $N = \bigcap_{i=1}^n \ker \tilde{\phi}_i$. Then $X = M \oplus N$:

- $x \in X \Rightarrow x = \sum_{i=1}^n \tilde{\phi}_i(x) x_i + \underbrace{\left(x - \sum_{j=1}^n \tilde{\phi}_j(x) x_j\right)}_{\text{put } y}$

$\tilde{\phi}_i(y) = \tilde{\phi}_i(x) - \sum_j \tilde{\phi}_j(x) \underbrace{\tilde{\phi}_i(x_j)}_{\delta_{ij}} = \tilde{\phi}_i(x) - \tilde{\phi}_i(x) = 0$

so $y \in N$

- $x \in M \cap N \Rightarrow x = 0$ by writing $x = \sum \alpha_i x_i$

$\alpha_i = \phi_i(x) \underset{x \in N}{=} 0$ ∎