Uniform boundedness principle (§ 23.3)

Th'm (Uniform Boundedness Principle for Banach spaces)

X, Y: Banach spaces, $\mathcal{A} \subset L(X, Y)$

Suppose $\forall x \in X \ \sup_{T \in \mathcal{A}} \|Tx\| < \infty$

Then $\sup_{T \in \mathcal{A}} \|T\| < \infty$ i.e. $\|Tx\|$ is uniformly bounded on $\{x \in X : \|x\| \le 1\}$

Idea: Put $E_\alpha = \{ x \in X : \sup_{T \in \mathcal{A}} \|Tx\| \le \alpha \} \underset{\text{closed}}{\subset} X$

$\bigcap_{T \in \mathcal{A}} \underbrace{\{x : \|Tx\| \le \alpha\}}_{\text{closed}}$

Assumption: $X = \bigcup_{n=1}^{\infty} E_n$.

St. 1 by Baire's category theorem $\exists x \in X, \varepsilon > 0$: $B^X(x, \varepsilon) \subset E_n$

otherwise $E_n^c = X \setminus E_n$ will be open dense

$\underset{\text{Baire}}{\Rightarrow} \bigcap_{n=1}^{\infty} E_n^c$ will be still dense.

St. 2 this implies $B^X(0, \varepsilon) \subset E_{2n}$ (cf. Step 2 of OMT Oct. 12)

St. 3 i.e. $\|x\| < \varepsilon \Rightarrow \sup_{T \in \mathcal{A}} \|Tx\| \le 2n$.

we get $\sup_{T \in \mathcal{A}} \|T\| = \sup_{T \in \mathcal{A}, \|x\| \le 1} \|Tx\| \le \frac{2n}{\varepsilon}$ $\blacksquare$

More precise form: "condensation of singularities"

if $\sup_{T \in \mathcal{A}} \|T\| = \infty$ then $\exists x$ s.t. $\sup_{T \in \mathcal{A}} \|Tx\| = \infty$

in fact such x's form a "residual set". ("large")

Def X: topological space

- $A \subset X$ is <u>nowhere dense</u> if $\overset{\circ}{\overline{A}} = \phi$ ($\circ$ ← interior, $\overline{\ }$ ← closure)

i.e. $\nexists \phi \ne U \underset{\text{open}}{\subset} X$ $U \subset \overline{A}$, equivalently

$\forall \phi \ne U \underset{\text{open}}{\subset} X$ $A \cap U$ not dense in U.

- $B \subset X$ is <u>meager</u> (or meagre, a set of 1st cat.)

if $B = \bigcup_{i=1}^{\infty} A_i$ A_i nowhere dense

- $C \subset X$ is <u>residual</u> (or comeager) if

$C^c = X \setminus C$ is meager. ($C = \bigcap_{i=1}^{\infty} D_i$, $\overset{\circ}{D_i}$ dense)

Ex. $\mathbb{Q} \subset \mathbb{R}$ is meager, $\therefore$ $\mathbb{R} \setminus \mathbb{Q}$ residual.

but : $\mathbb{R}^{\oplus\infty}$ itself is meager

Th'm (Banach - Steinhaus, UBP, 25.27)

X, Y : normed vector spaces, $\mathcal{A} \subset L(X, Y)$

Put $F = \{ x \in X : \sup_{T \in \mathcal{A}} \| Tx \| < \infty \}$, $R = X \setminus F$
$= \{ x : \sup_{T \in \mathcal{A}} \| Tx \| = \infty \}$

Then 1. F not meager $\Rightarrow \sup_{T \in \mathcal{A}} \| T \| < \infty$

2. if X is a Banach space,

F not meager $\Longleftrightarrow \sup_{T \in \mathcal{A}} \| T \| < \infty$

$F = X \Longleftarrow$

2'. if X is a Banach space,

R is residual $\Longleftrightarrow \sup_{T \in \mathcal{A}} \| T \| = \infty$

($\Leftarrow$: concentration of singularities)

Proof : 1 : put $E_\alpha = \{ x : \sup_{T \in \mathcal{A}} \| Tx \| \le \alpha \}$

closed in X

So $F = \bigcup_{n=1}^{\infty} E_n$; F not meager $\Rightarrow \exists n \quad \mathring{E}_n \neq \emptyset$

Claim 1 : Take x, δ s.t. $B^X(x, \delta) \subset E_n \Rightarrow \overline{B^X(x, \delta)} \subset E_n$

i.e. $\| x' \| \le \delta \Rightarrow x + x' \in E_n$ $(\| T(x + x') \| \le n)$

We have $\| T \underbrace{x'}_{x + x' - x} \| \le \| T(x + x') \| + \| Tx \| \le 2n$

By rescaling $\| x' \| \le 1 \Rightarrow \| Tx \| \le \frac{2n}{\delta}$. i.e. $\| T \| \le \frac{2n}{\delta}$.

so $\sup_{T \in \mathcal{A}} \| T \| \le \frac{2n}{\delta}$

Claim 2 : we need "$\Leftarrow$" ; X is not meager

by Baire's cat. thm

Claim 2' : take contrapositive of 2 $\blacksquare$