Application of uniform boundedness principle
(condensation of singularities)

Want to understand divergence of partial sums of Fourier series

$T = \{ w = e^{i\theta} : \theta \in \mathbb{R} \} \subset \mathbb{C}$, consider measure μ characterized by $\int f \, d\mu = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) \, d\theta$

$\phi_n(w) = w^n = e^{in\theta}$ for $n \in \mathbb{Z}$ form an orthonormal basis of $L^2(T, \mu)$ i.e.,

- $(\phi_m, \phi_n)_{L^2(T,\mu)} = \int \phi_m \overline{\phi_n} \, d\mu = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(m-n)\theta} d\theta$
$= \delta_{m,n}$.

- $f \in L^2(T, \mu) \Rightarrow f = \sum_{n=-\infty}^{\infty} \alpha_n \phi_n$
$\alpha_n = (f, \phi_n)_{L^2(T,\mu)}$

To pe precise; put $S_N(f) := \sum_{n=-N}^{N} \alpha_n \phi_n$

then $\| f - S_N(f) \|_{L^2} \to 0 \quad (N \to \infty)$

But no guarantee that $\| f - S_N(f) \|_\infty \to 0$.
in fact, this can fail for many f.

Lem. $f \mapsto S_N(f)$ is given by
$$S_N(f)(z) = \int f(w) D_N(z \bar{w}) \, d\mu(w)$$
(write $(f * D_N)(z)$, convolution by D_N)

for $D_N(e^{i\theta}) = \dfrac{\sin(N+\frac{1}{2})\theta}{\sin \frac{1}{2}\theta}$ (Dirichlet kernel)

Proof Put $D_N^\circ = \sum_{n=-N}^{N} \phi_n$ then $D_N^\circ(z\bar{w}) = \sum_{n=-N}^{N} \phi_n(z) \overline{\phi_n(w)}$

from $\phi_n(z\bar{w}) = z^n \bar{w}^n = \phi_n(z) \overline{\phi_n(w)}$

So $\int f(w) D_N^\circ(z\bar{w}) \, d\mu(w) = \sum_{n=-N}^{N} (f, \phi_n) \cdot \phi_n(z) = S_N(f)(z)$

It remains to prove $D_N^0(e^{i\theta}) = \dfrac{\sin(N+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}$

$$\underbrace{(e^{\frac{i}{2}\theta} - e^{-\frac{i}{2}\theta})}_{2i\sin\frac{1}{2}\theta}\underbrace{(e^{Ni\theta} + e^{(N-1)i\theta} + \cdots + e^{-Ni\theta})}_{D_N^0(e^{i\theta})} = \underbrace{e^{(N+\frac{1}{2})i\theta} - e^{-(N+\frac{1}{2})i\theta}}_{2i\sin(N+\frac{1}{2})\theta}$$

$\blacksquare$

Prop (25.30) given $z \in T$, $\exists$ residual set $R_z \subset C(T)$ s.t. $\sup\limits_{N=1,2,\cdots} |s_N(f)(z)| = \infty$ for $f \in R_z$

Rem. $C(T)$ complete for $\|-\|_\infty \Rightarrow R_z$ dense.

(also $z_1, z_2, \cdots \in T \Rightarrow \bigcap\limits_{n=1}^{\infty} R_{z_n}$ dense)

$f \in R_z \Rightarrow s_N(f)$ cannot converge to f for $\|-\|_\infty$.

Proof Step 1 reduction to $z=1$

consider coordinate change $e^{i\theta} \to e^{i(\theta-\theta_0)}$

Step 2 the functional $\Lambda_N : C(T) \to \mathbb{C}$

$\Lambda_N(f) = s_N(f)(1)$ satisfies $\|\Lambda_N\| = \int |D_N(m)|\, d\mu(m)$

Recall $\|\Lambda_N\| = \sup\limits_{f \in C(T), \|f\|_\infty \le 1} |\Lambda_N(f)|$

We have $\Lambda_N(f) = \int f(m) D_N(\bar{m})\, d\mu(m)$

So $f(m) = \mathrm{sign}(D_N(\bar{m}))$ will achieve the equality for $\int f \cdot D_N\, d\mu \le \|f\|_\infty \cdot \|D_N\|_1$

To approximate $\mathrm{sign}\, D_N(\bar{m})$ in $C(T)$,

take $f_\varepsilon(m) = \dfrac{D_N(\bar{m})}{\sqrt{D_N(\bar{m})^2 + \varepsilon}}$

Step 3 $\|\Lambda_N\| \to \infty$ as $N \to \infty$.

$$\int |D_N(\bar{m})|\, d\mu(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(e^{-i\theta})\, d\theta$$

$$\ge \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|\sin(N+\frac{1}{2})\theta|}{|\frac{1}{2}\theta|}\, d\theta = \frac{2}{\pi}\int_0^{\pi} \sin(N+\tfrac{1}{2})\theta \frac{d\theta}{\theta}$$

$$\underset{y=(N+\frac{1}{2})\theta}{=} \frac{2}{\pi}\int_0^{(N+\frac{1}{2})\pi} \underbrace{|\sin y|}\, \frac{dy}{y}.$$

$\exists\, C > 0$ s.t. $\displaystyle\int_0^{N+\frac{1}{2}} \sin y \,\frac{dy}{y} \geq C \sum_{k=1}^{N} \frac{1}{k}$

by $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} = \infty$ we get $\|\Lambda_N\| \to \infty$.

Step 4 Claim ; $R_1 = \{ f \in C(T) : \sup_{N=1,2,\dots} |\Lambda_N(f)| = \infty \}$

is residual in $C(T)$

this follows from Step 3 and Banach-Steinhaus thm $\square$