Goal : given top. sp. X, understand rel. betw.

- measures μ on $(X, \mathcal{B}_X)$;

  $\mathcal{B}_X$ collection of Borel sets

  i.e. $\mathcal{B}_X = \sigma$-alg. generated by the coll

  $$\{ U \subset X : \text{open} \}.$$

  extra regularities ...

- functionals ϕ on space(s) of cont. funcs

  $\{ f : X \to \mathbb{R}$ cont, extra conditions ... $\}$

$\mu \rightsquigarrow \phi(f) = \int f \, d\mu$

- conditions on f should make the int. well-def'ed in $\mathbb{R}$.
- can we expect boundedness of ϕ ?

  other properties ?

$\phi \rightsquigarrow \mu(A) =$ "$\lim_{f \to 1_A} \phi(f)$."

- what kind of regularity (on μ) should we expect if this makes sense ?
- what do we need on ϕ to ensure that this def's a measure on $(X, \mathcal{B}_X)$?

Reasonable framework

1 X : locally compact Hausdorff space.

  i.e. Hausdorff & $\forall x \in X$ has a cpt neighbourhood.

Ex. $[0,1]$, $\{0,1\}^{\mathbb{N}}$. cpt metrizable

  $\mathbb{R}$ metrizable

  $\{0,1\}^S$ for $|S| > \aleph_0$ non-metrizable

  but could be separable ($\exists$ dense cbl set).

Non-ex : ∞-dim Banach sp.

function space : $C_c(X)$ (or $C_c(X, \mathbb{R})$)

$\{f : X \to \mathbb{R}$, continuous, compact support $\}$

i.e. $\{x \in X : f(x) \neq 0\}$ has cpt closure. (call it $\mathrm{supp}(f)$)

functional : positive functionals

Def (28.8) $\phi : C_c(X) \to \mathbb{R}$ linear func. is positive

if $\phi(f) \geq 0$ for $f \geq 0$.

Rem. this makes sense for funcs. $C_c(X, \mathbb{C}) \to \mathbb{C}$ also.

Prop (28.9) ϕ pos. as above, $K \subset X$ cpt.

Then $\exists\, C_K > 0$ s.t. $|\phi(f)| \leq C_K \|f\|_\infty$

for all $f \in C_c(X)$ with $\mathrm{supp}(f) \subset K$.

Same w/ funcs on $C_c(X, \mathbb{C})$

Proof. Take $g \in C_c(X)$ s.t.

- $0 \leq g(x) \leq 1$ for $x \in X$
- $g(x) = 1$ for $x \in K$.

(possible by Tietze ext or Urysohn's lemm.)

Claim : $C_K = \phi(g)$ will do. i.e.

$|\phi(f)| \leq \phi(g) \cdot \|f\|_\infty$.

St. 3 / $\mathbb{C}$ Choose $\alpha \in \mathbb{C}$, $|\alpha| = 1$ s.t.

$\phi(\alpha f) = |\phi(f)|$

Note $\|\alpha f\|_\infty = \|f\|_\infty$ so we may assume

$\phi(f) = |\phi(f)|$ $(= \phi(\mathrm{Re}\, f)$ for $C_c(X, \mathbb{C}))$.

St. 1 $\|f\|_\infty \cdot g \pm f \geq 0$. $f \in C_c(X, \mathbb{R})$, $\mathrm{supp}(f) \subset K$

St. 2 $|\phi(f)| \leq \|f\|_\infty \cdot \phi(g)$ from St. 1

$\phi|_{C_c(X, \mathbb{R})}$ is pos. as real lin. form.

regularity for measures

Def (28.12) μ : measure on $(X, \mathcal{B}_X)$, $A \in \mathcal{B}_X$

- μ is <u>inner regular</u> on A if

$$\mu(A) = \sup_{K \subset A, \text{ cpt subset of } X} \mu(K)$$

- μ is <u>outer regular</u> on A if

$$\mu(A) = \inf_{U \supset A, \text{ open subset of } X} \mu(U)$$

Def. (28.12 & 13)

<u>regular Borel measure</u> on X : μ on $(X, \mathcal{B}_X)$

which is inner & outer reg. for all $A \in \mathcal{B}_X$
($\mu(K) < \infty$ for cpt K.)

<u>Radon measure</u> on X : μ on $(X, \mathcal{B}_X)$

which is - inner reg. on all open sets of X
- outer reg. on all $A \in \mathcal{B}_X$
- $\mu(K) < \infty$ for cpt K.

Rem σ-fin. Radon meas is reg. (Borel.)

so for sep. metrizable loc. cpt. space (like $\mathbb{R}$)

two conditions are equivalent

(but not for $\{0,1\}^S$, $|S| > \aleph_0$, ...)

Riesz - Markov theorem (Riesz rep for measures)

X loc. cpt sp The corresp.

$\{\mu$; Radon meas on $X\}$

$\downarrow$ $\mu \mapsto \phi_\mu(f) = \int f \, d\mu$

$\{\phi$: pos. func. on $C_c(X)\}$

is well-defied and bijective.