Th'm (Riesz - Markov 28.16)

X : loc cpt Hausdorff top sp Then we have a bij correspondence between

- Radon measures μ on X ; $\mu(K) < \infty$ for cpt $K \subset X$
  $\mu(U) = \sup_{K \subset U \text{ cpt}} \mu(K)$ for open U
  $\mu(A) = \inf_{A \subset U \text{ open}} \mu(U)$ for Borel A
- positive linear functionals ϕ on $C_c(X, \mathbb{R})$;
  $\phi(f) \geq 0$ for $f \geq 0$

$\mu \rightsquigarrow \phi(f) = \int f \, d\mu$. $\phi \rightsquigarrow \mu(U) = \sup_{0 \leq f \leq 1, \, f \in C_c(U)} \phi(f)$
for open U .

Part 1 $\phi \rightsquigarrow \mu$ is well defi'ed as measure by
$\mu(A) = \inf_{A \subset U \text{ open}} \mu(U)$, $\mu(U)$ from ϕ as above
for $A \in \mathcal{B}_X$

Step 1. Put $\mu^*(Y) = \inf \{ \sum_{i=1}^{\infty} \mu(U_i) : U_i$ open, $Y \subset \bigcup_{i=1}^{\infty} U_i \}$
for $Y \subset X$. Then μ^* is an outer measure on X

i.e. $\mu^*(\phi) = 0$, $Y \subset Z \Rightarrow \mu^*(Y) \leq \mu^*(Z)$, $\mu^*(\bigcup_{i=1}^{\infty} Y_i) \leq \sum_{i=1}^{\infty} \mu^*(Y_i)$

Step 2 $\mathcal{M} = \{ A \subset X : \mu^*(Y) = \mu^*(Y \cap A) + \mu^*(Y \setminus A)$
for all $Y \subset X \}$
is a σ- alg , and
μ^* def's a measure on $\mathcal{M}$ (Th'm 31.17)

Step 3 $\mathcal{M}$ contains all open sets ($\Rightarrow \mathcal{M} \supset \mathcal{B}_X$,
μ is a measure on $\mathcal{B}_X$)

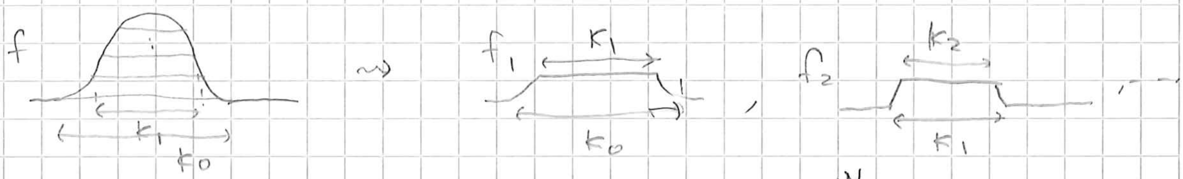
Part 2 For $\phi \rightsquigarrow \mu$ in Part 1, $\phi(f) = \int f \, d\mu$

for $f \in C_c(X)$, $0 \le f$

By rescaling we'll assume $\|f\|_\infty = 1$

Fix $N \in \mathbb{N}$ (large). Put $K_n = \{x \in X : f(x) \geq \frac{n}{N}\}$

and $K_0 = \text{supp}(f)$ (so $K_n = \emptyset$ for $n > N$)



so $\frac{1}{N} 1_{K_i} \le f_i \le \frac{1}{N} 1_{K_{i-1}}$, $f = \sum_{i=1}^{N} f_i$

Step 1 $\left| \int f_i \, d\mu - \phi(f_i) \right| \le \frac{1}{N} \mu(K_{i-1} \setminus K_i)$

$$\mu(K_i) \underbrace{\le \phi(N f_i) \le}_{\text{outer reg.}} \mu(K_{i-1})$$

and $\frac{1}{N}\mu(K_i) \le \int f \, d\mu \le \frac{1}{N} \mu(K_{i-1})$

Step 2 $\left| \int f \, d\mu - \phi(f) \right| \le \frac{1}{N} \mu(K_0)$

take sum of terms in Step 1

Part 3. μ is inn. reg. on open sets ($\Rightarrow \mu$ Radon)

Step 1 $\mu(U) < \infty \Rightarrow \forall \varepsilon \ \exists K \subset U$ cpt, $\mu(K) \geq \mu(U) - \varepsilon$

Fix $\varepsilon > 0$. From $\mu(U) = \sup_{f \in C_c(U), \, 0 \le f \le 1} \phi(f)$

$\exists f$ $\phi(f) \geq \mu(U) - \varepsilon$. Then $K = \text{supp}(f)$

satisfies $K \subset U$ cpt, $1_K \geq f$.

$$\mu(K) = \int 1_K \, d\mu \geq \int f \, d\mu \underset{\text{Part 2}}{=} \phi(f) \geq \mu(U) - \varepsilon.$$

Step 2 $\mu(U) = \infty \Rightarrow \exists K_1, K_2, \ldots \subset U$ cpt

$\mu(K_i) \to \infty$.

Take $f_1, f_2, \ldots \in C_c(X)$ $\text{supp}(f_i) \subset U$, $0 \le f_i \le 1$

$\phi(f_i) \to \infty$. Then $K_i = \text{supp}(f_i)$ will do

Part 4 For $\mu \rightsquigarrow \phi(f) = \int f \, d\mu$

$$\mu(U) = \sup_{f \in C_c(U),\, 0 \le f \le 1} \phi(f) \quad \text{for } U \subset X \text{ open.}$$

(then outer reg. says ϕ "remembers" μ)

Fix $\varepsilon > 0$. By inner regularity

$\exists\, K \subset U$ cpt. $\mu(K) \geq \mu(U) - \varepsilon$.

Urysohn's Lem $\Rightarrow \exists\, f \in C_c(X)$ s.t. $f|_K = 1_K$,

$\operatorname{supp}(f) \subset U$. Then $\phi(f) \geq \mu(K) \geq \mu(U) - \varepsilon$

$f \in C_c(U)$, $0 \le f \le 1 \Rightarrow \phi(f) = \int f \, d\mu \le \int 1_U \, d\mu = \mu(U)$

$\square$

Rem. Consider norm $\|f\|_\infty$ on $C_c(X, \mathbb{R})$

$\|\phi\| = \mu(X)$ up to $\phi_{\text{pos}} \longleftrightarrow \mu$ Radon

Key : $\mu(X) = \sup_{K \subset X \text{ cpt}} \mu(K) \geq \sup_{f \in C_c(X),\, \underline{0 \le f \le 1}} \phi(f)$.

$\Rightarrow \|f\|_\infty \le 1$.

Part 1 Step 3. $U \in \mathcal{M}$ for open U

3-1 Y open $\Rightarrow \mu^*(Y) = \mu^*(Y \cap U) + \mu(Y \setminus U)$

$Y \cap U$ open $\Rightarrow$ we can choose $f \in C_c(Y \cap U)$

$0 \le f \le 1$, $\phi(f) \sim \mu(Y \cap U) = \mu^*(Y \cap U)$

$Y \setminus U \subset Y \setminus \operatorname{supp}(f) = V$ open

Choose $g \in C_c(V)$, $0 \le g \le 1$, $\phi(g) \sim \mu(V)$

Then $0 \le f + g \le 1$, $f + g \in C_c(Y)$

So $\mu^*(Y) = \mu(Y) \geq \phi(f) + \phi(g)$.

Take $\sup_{g \text{ as above}} \rightsquigarrow \mu^*(Y) \geq \phi(f) + \mu(V) \geq \phi(f) + \mu^*(Y \setminus U)$

Take $\sup_{f \text{ as above}}$ $\mu(Y) \geq \mu(Y \cap U) + \mu(Y \setminus U)$

3-2 $\mu^*(Y) = \mu^*(Y \cap U) + \mu^*(Y \setminus U)$ for $Y \subset X$

$V \supset Y$ open $\Rightarrow \mu(V) \geq \mu^*(V \cap U) + \mu^*(V \setminus U)$

$\geq \mu^*(Y \cap U) + \mu^*(Y \setminus U)$

$\mu^*(Y) = \inf_{\text{such } V} \mu(V) \Rightarrow \mu^*(Y) \geq \mu^*(Y \cap U) + \mu^*(Y \setminus U)$

conv. ineq. from subadditivity.