Martingale convergence.

$(\Omega, \mathcal{F}, \mu)$ probability space

$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ σ-subalgs of $\mathcal{F}$ s.t. $\mathcal{F} = \sigma(\bigcup_{i=1}^{\infty} \mathcal{F}_i)$

Recall

- $\bigcup_{i=1}^{\infty} L^p(\Omega, \mathcal{F}_i, \mu)$ dense in $L^p(\Omega, \mathcal{F}, \mu)$

  for $1 \le p < \infty$. (Sept 28 ; Th'm 22.4)

- conditional expectation

  $E_{\mathcal{F}_i} : L^q(\Omega, \mathcal{F}, \mu) \to L^q(\Omega, \mathcal{F}_i, \mu)$ $\quad (1 < q \le \infty)$
  $f \mapsto E[f \mid \mathcal{F}_i]$

  from L^p-L^q duality (Sep 29)

Def. a <u>martingale</u> is a sequence of random variables $X_1, X_2, \dots$ on Ω s.t.

- X_i is $\mathcal{F}_i$-measurable, say $X_i \in L^q(\Omega, \mathcal{F}_i, \mu)$
- $E[X_i \mid \mathcal{F}_j] = X_j$ for all $j < i$

Ex. fair gamble, profit from each game modelled by rand var. Y on $(\Omega', \mathcal{F}', \nu)$ s.t.

$E[Y] = 0$ $\rightsquigarrow$ infinitely many times

modelled by $X_i = Y^{(1)} + \cdots + Y^{(i)}$ (profit after i)

on $\Omega = \prod_{i=1}^{\infty} \Omega'$, $\mathcal{F}_i = \{ A \times \prod_{j=i+1}^{\infty} \Omega' : A \in \underbrace{\mathcal{F}' \otimes \cdots \otimes \mathcal{F}'}_{i \text{ times}} \}$

We want to say : $\exists\, X \in L^q(\Omega, \mathcal{F}, \mu)$ representing the "limit" of $(X_i)_{i=1}^{\infty}$.

Th'm (Doob) $(X_i)_i$ martingale in $L^\infty(\Omega, \mathcal{F}, \mu)$

Suppose $\sup \|X_i\| < \infty$

Then $\exists\, X \in L^\infty(\Omega, \mathcal{F}, \mu)$, $X_i(\omega) \to X(\omega)$ a.e.

We will use Doob's inequality: if $(X_i)_i$ is an L^1-martingale, $\forall \varepsilon > 0$

$$\varepsilon \cdot \underbrace{\mu(\{\omega \in \Omega : \sup |X_i(\omega)| > \varepsilon\})}_{\text{"}\mathbb{P}[X_i > \varepsilon \text{ for some } i]\text{"}} \le \sup \|X_i\|_1.$$

Candidate for $X = \lim X_i$: (in $L^\infty(\Omega, \mathcal{F}, \mu)$)

by L^1-L^∞ Duality, enough to specify $\phi(f) = \int X(\omega) f(\omega) d\mu(\omega)$ for $f \in L^1(X, \mathcal{F}, \mu)$.

For $f \in L^1(X, \mathcal{F}_i, \mu)$ put $\phi(f) = \int X_i(\omega) f(\omega) d\mu(\omega)$

- well-defied on $\bigcup L^1(\Omega, \mathcal{F}_i, \mu)$
- bdd in $\|f\|_1$ (by $(\sup \|X_i\|_\infty) \times \|f\|_1$)

$\rightsquigarrow$ extends to a bdd lin functional on $L^1(\Omega, \mathcal{F}, \mu)$

By construction $E[X \mid \mathcal{F}_i] = X_i$

Checking $X_i(\omega) \to X(\omega)$ a.e. ; fix $\varepsilon > 0$.

take $Y = Y^\varepsilon \in L^\infty(\Omega, \mathcal{F}, \mu) \cap \left(\bigcup_{i=1}^\infty L^1(\Omega, \mathcal{F}_i, \mu)\right)$

s.t. $\|X - Y\|_1 < \varepsilon^2$. (possible by density)

Step 1 $Y_i = E[Y \mid \mathcal{F}_i]$ satisfy $Y_i(\omega) \to Y(\omega)$ $\forall \omega \in \Omega$.

$Y \in L^1(\Omega, \mathcal{F}_j, \mu) \Rightarrow Y_k = Y$ for $k \ge j$

Step 2 $T_\varepsilon = \{\omega \in \Omega : \sup_i |X_i(\omega) - Y_i(\omega)| > \varepsilon\}$

Satisfies $\mu(T_\varepsilon) < \varepsilon$. ($\mathbb{P}[|X_i - Y_i| > \varepsilon$ for some $i]$)

- $\|X_i - Y_i\|_1 \le \|X - Y\|_1$ from $\|E_{\mathcal{F}_i}\| \le 1$
- Doob's ineq. for $X_i - Y_i$.

Start moving ε.

Step 3 $A = \{\omega \in \Omega : \omega \in T_{2^{-n}}$ holds for ∞-many $n\}$ is μ-null.

Put $f = \sum_{n=1}^\infty 1_{T_{2^{-n}}}$ so $A = \{\omega : f(\omega) = \infty\}$

$\int f \, d\mu = \sum \mu(T_{2^{-n}}) < \infty$ implies $\mu(A) = 0$

Step 4 Suppose $X_i(\omega)$ is not convergent.

then $\exists \varepsilon_0 \ \forall \varepsilon < \varepsilon_0$; $|X_i(\omega) - Y_i^{\varepsilon}(\omega)| > \varepsilon$

for infinitely many i.

Pick ε_0 s.t. $\exists$ s, t $|s - t| > \varepsilon_0$,

$\exists$ subseqs $X_{i_k}(\omega) \to s$, $X_{i'_k}(\omega) \to t$

$Y_i^{\varepsilon}(\omega)$ is convergent $\Rightarrow$ claim.

Step 5 $X_i(\omega) \to X(\omega)$ a.e.

if $X_i(\omega) \not\to X(\omega)$, Step 4 implies $\omega \in A$
null by Step 3.