Fourier transform (following Folland §8.3)

x, ξ : variable on $\mathbb{R}^n$.

$f(x)$ function with "good" regularity
(cplx valued)

$\rightsquigarrow$ $\mathcal{F}[f](\xi) = \int f(x) e^{-2\pi i x \cdot \xi} dx$ Fourier trans.

also write $\hat{f}(\xi)$

other conventions : $\int f(x) e^{2\pi i x \cdot \xi} dx$, $\frac{1}{(2\pi)^{n/2}} \int f(x) e^{i x \xi} dx$

inverse Fourier transform : given $g(\xi)$

$$\mathcal{F}^{-1}[g](x) = \int g(\xi) e^{2\pi i x \cdot \xi} d\xi$$

also write $\check{g}(x)$.

(We will work out regularity later.)

Formal consequences of formula :

Prop.

1) $\mathcal{F}[f_1 * f_2] = \hat{f}_1 \hat{f}_2$ for

$$f_1 * f_2 (x) = \int f_1(y) f_2(x-y) dy = \int f_1(x-y) f_2(y) dy$$

2) $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ multiindex

$$\mathcal{F}[\partial^\alpha f](\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi), \quad \mathcal{F}[(-2\pi i x)^\alpha f] = (\partial^\alpha \hat{f})(\xi)$$

for $\partial^\alpha f(x) = \frac{\partial^{|\alpha|} f}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}}(x)$, $(2\pi i \xi) = (2\pi i)^{|\alpha|} \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$

$|\alpha| = \alpha_1 + \cdots + \alpha_n$.

cf. Folland Thm 8.22 ; Driver Thm 34.3.

1. is for $f_1, f_2 \in L^1(\mathbb{R}^n)$, use Fubini to justify

$$\int dx \int dy \; f_1(x-y) f_2(y) e^{-2\pi i x \xi}$$
$$= \int dy \int dx \; f_1(x-y) e^{-2\pi i (x-y)\xi} f_2(y) e^{-2\pi i y \cdot \xi}$$

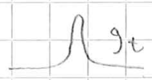
2 is for $\partial^\beta f \in L^1(\mathbb{R}^n)$, $\partial^{\beta'} f \in C_0(\mathbb{R}^n)$ $\beta \le \alpha$, $\beta' < \alpha$
$x^\beta f \in L^1(\mathbb{R}^n)$

Schwartz class $\mathcal{S}(\mathbb{R}^n) = \{ f(x) : \|f\|_{N,\alpha} < \infty \quad \forall N, \alpha$

$$\|f\|_{N,\alpha} = \sup_x (1+|x|)^N |\partial^\alpha f(x)| \}$$

Prop (2) implies $f \in \mathcal{S}(\mathbb{R}^n) \Rightarrow \hat{f} \in \mathcal{S}(\mathbb{R}^n)$

Th'm (F. 8.26, D. 34.11) $\mathcal{F}^{-1}[\hat{f}] = f$ if $f, \hat{f} \in L^1$.
$(\mathcal{F}[\check{f}] = f$ also$)$.

Outline $g_t(x) = t^{-n} \exp(-\pi |\frac{x}{t}|^2)$  $t \ll 1$

- $f * g_t \to f$ in L^1
- $f * g_t = \mathcal{F}^{-1}[e^{-\pi|t\xi|^2} \hat{f}(\xi)]$
- $t \to 0$ gives $\mathcal{F}^{-1}[e^{-\pi|t\xi|^2} \hat{f}(\xi)] \to \mathcal{F}^{-1}[\hat{f}]$

Th'm (Plancherel, F. 8.29, D 34.11)

- $\|\hat{f}\|_2 = \|f\|_2$ for $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

$(\Rightarrow \mathcal{F}, \mathcal{F}^{-1}$ extends to isomet. transform on $L^2(\mathbb{R}^n))$

$(f_1, f_2)_{L^2} = (\hat{f}_1, \hat{f}_2)_{L^2}$

Pf. $(f_1, f_2)_{L^2} = (f_1 * g)(0)$ for $g(x) = \overline{f_2(-x)}$

- $f_1 * g = \mathcal{F}^{-1}\mathcal{F}[f_1 * g] = \mathcal{F}^{-1}[\hat{f}_1 \hat{g}]$

$\Rightarrow f_1 * g(0) = \int \hat{f}_1(\xi) \hat{g}(\xi) d\xi = (\hat{f}_1, \bar{\hat{g}})_{L^2}$.

- $\hat{g}(\xi) = \overline{\hat{f}_2(\xi)}$

Hausdorff-Young inequality $1 \le p \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$

$\|\hat{f}\|_q \le \|f\|_p$.