Modelling stochastic processes

Problem $(X_\alpha)_{\alpha \in A}$ family of random vars.

i.e. prob. meas. μ_α on $\mathbb{R}$ is given for each α ; relations for $(X_\alpha)_{\alpha \in F}$ $F \subset\subset A$

$\rightsquigarrow$ construct one prob. sp. $(\Omega, \mathcal{F}, \mathbb{P})$ and funcs. $f_\alpha : \Omega \to \mathbb{R}$ representing X_α.

$(f_\alpha)_* \mathbb{P} = \mu_\alpha$, f_α same rel. as X_α

Example : - sequence $X_1, X_2, \cdots$ some quantity at "discrete time"

- $(X_t)_{t \in \mathbb{R}}$ some quantity at cont. time

Wiener process $(X_t)_{t \geq 0}$

- $X_0 = 0$
- $0 \leq s < t \Rightarrow X_t - X_s$ has the normal distr. with mean 0, variance $t-s$.

  i.e. $\mathbb{E}[(X_t - X_s)^k] = \int_{-\infty}^{\infty} x^k \frac{1}{\sqrt{\pi(t-s)}} e^{-\frac{1}{2}\frac{x^2}{t-s}} dx$
- $0 \leq t_0 < t_1 < \cdots < t_n \Rightarrow X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \cdots, X_{t_n} - X_{t_{n-1}}$ mutually indep

Back to general setting

relation for $(X_\alpha)_{\alpha \in F}$ $F = (\alpha_1, \cdots, \alpha_k)$ $\alpha_i \in A$

$\leftrightarrow$ joint distribution μ_F ; $\mu(E) = \mathbb{P}[(X_{\alpha_1}, \cdots, X_{\alpha_k}) \in E]$ for $E \subset \mathbb{R}^k$

Consistency conditions

(C1) $\sigma \in S_k \Rightarrow \sigma_* \mu_{\sigma F} = \mu_F$

$\mathbb{P}[(X_{\alpha_{\sigma(1)}}, \cdots, X_{\alpha_{\sigma(k)}}) \in \sigma F] = \mathbb{P}[(X_1, \cdots, X_k) \in F]$

(C2) $k < n \Rightarrow \mu_{(\alpha_1, \cdots, \alpha_k)}(E) = \mu_{(\alpha_1, \cdots, \alpha_n)}(E \times \mathbb{R}^{n-k})$

Th'm (F. 10.18) If a family μ_F of prob meas. on $\mathbb{R}^k$ for $F = (\alpha_1, \cdots, \alpha_k)$ $\alpha_i \in A$, $k \in \mathbb{N}$

1 (cont.)

satisfies (C1) & (C2) $\Rightarrow \exists$ cpt Hausdorff sp. Ω, Radon meas. μ, funcs. $f_\alpha : \Omega \to \mathbb{R}$ s.t. $\mu_{(\alpha_1, \dots, \alpha_k)}$ is the joint distr. of $f_{\alpha_1}, \dots, f_{\alpha_k}$

Idea:

St. 1 $\Omega = \overline{\mathbb{R}}^A$ for $\overline{\mathbb{R}} = [-\infty, \infty]$ as top. sp
$(\simeq [-1, 1]$ by $x \mapsto \frac{x}{1+|x|})$

St. 2 $f_\alpha : \Omega \to \overline{\mathbb{R}}$ proj. to α-th factor.

$\rightsquigarrow$ Ω is a cpt Hausdorff sp. by Tychonoff's thm
(metrizable $\Leftrightarrow$ A at most ctbl), f_α cont.

St. 3 giving μ $\underset{\text{Riesz-Markov}}{\Leftrightarrow}$ giving $\underset{\text{positive}}{\phi : C(\Omega) \to \mathbb{R}}$

St. 4 $C_{fin}(\Omega) = \{ f \in C(\Omega),\ f(x_\alpha) = f(x_{\alpha_1}, \dots, x_{\alpha_k})$ for some $(\alpha_1, \dots, \alpha_k)$, $\alpha_i \in A \}$

$\rightsquigarrow$ $C_{fin}(\Omega)$ is a dense subalg of $C(\Omega)$
(Stone-Weierstrass)

$f = f(x_{\alpha_1}, \dots, x_{\alpha_k}) \in C_{fin}(\Omega)$ $\rightsquigarrow$ put $\phi(f) = \int f \, d\mu_F$
$F = (\alpha_1, \dots, \alpha_k)$

This is well defied by C1 & C2., positive.

St. 5 ϕ extends to $C(\Omega)$. $\because$ $|\phi(f)| \leq \|f\|_\infty$

St. 6 joint dist of $f_{\alpha_1}, \dots, f_{\alpha_k}$ for μ is $\mu_{(\alpha_1, \dots, \alpha_k)}$

$g \in C(\overline{\mathbb{R}}^k) \Rightarrow \underset{\int g \, d(f_{\alpha_1}, \dots, f_{\alpha_k})_* \mu}{\underset{\wr}{g \circ (f_{\alpha_1}, \dots, f_{\alpha_k})}} = \underset{\int g \, d\mu_F}{\underset{\wr}{g(x_{\alpha_1}, \dots, x_{\alpha_k})}}$. ∎