Convex sets and topology

Setting : X (rect) Banach space (or loc conv top sp.)

- $A \subset X$ is convex if $x, y \in A$, $0 \le t \le 1$

$\Rightarrow \quad tx + (1-t)y \in A$.

- weak topology on X : open sets are unions of finite intersections of

$$\mathcal{O}_{\varphi, U} = \{ x \in X : \varphi(x) \in U \}$$

for φ bounded functional on X

$U \subset \mathbb{R}$ open set

(Ex. $\{ x \in X : a_1 < \varphi_1(x) < b_1, \cdots, a_k < \varphi_k(x) < b_k \}$ is open for the wk top.)

Rem. $A \subset X$ open for the weak top. (wkly open)

$\Rightarrow$ open for the norm top. (strongly open)

but not $\Leftarrow$ when $\dim X = \infty$.

$\{ x \in X : \|x\| < 1 \}$ is not wkly open

Th'm. $A \subset X$ convex and norm closed

$\Rightarrow$ A is also weakly closed.

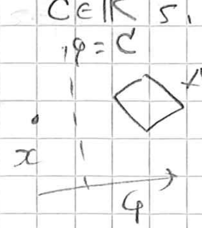
Ex. $\{ x \in X : \|x\| \le 1 \}$ is wkly closed

$$\overset{\text{Hahn-Ban.}}{\|} \quad \underbrace{\bigcap_{\varphi \in X^*}}_{\text{intersection preserves closedness.}} \underbrace{\{ x : |\varphi(x)| \le \|\varphi\| \}}_{\text{wkly closed}}$$

Key phenomena : $x \notin A \Rightarrow \exists\, \varphi \in X^*$, $c \in \mathbb{R}$ s.t.

$\varphi(x) < c \le \varphi(y) \quad (y \in A)$

i.e. φ separates x from A

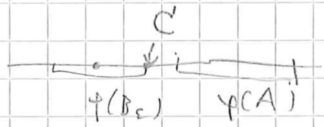


Proof of separation.

First reduction: $x = 0$: $y \rightsquigarrow y - x$.

Separation: - take $\varepsilon > 0$ s.t. $B_\varepsilon = \{x' : \|x'\| < \varepsilon\}$ does not intersect w/ A : possible b/c A is closed

Another red: enough to find φ s.t. $\varphi(x') < \varphi(y)$ for $x' \in B_\varepsilon$, $y \in A$.



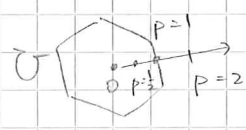
Fix $y_0 \in A$. Idea: find φ s.t. $\|\varphi\|$ small but $\varphi(y_0)$ larger enough.

$\Rightarrow$ $\varphi(B_\varepsilon)$ small

$y \in A \Rightarrow \varphi(y) \sim \varphi(y_0)$ far from $\varphi(B_\varepsilon)$

Control cond for φ: take $U = \{x' - y + y_0 : x' \in B_\varepsilon, y \in A\}$ convex, open, $y_0 \notin U$

Scale func $p(z) = \inf\{s > 0 : s^{-1} z \in U\}$



- p is a Minkowski functional $p(x+y) \leq p(x) + p(y)$, $p(tx) = t\,p(x)$, $t > 0$

$U = \{z \in X : p(z) < 1\}$

$Y = \mathbb{R} y_0 \subset X$ subsp. $\varphi(t y_0) = t$.

So $\varphi(y') \leq t \cdot m(y')$ on Y $(m(y_0) \geq 1)$

Hahn-Banach thm gives $\varphi \in X^*$, $\varphi(x) \leq t\, m(x)$.

$\varphi(x) < 1$ for $x \in U_\varepsilon$ neigh. of 0 means $\varphi \in X^*$.

$x' \in B_\varepsilon$, $y \in A \Rightarrow \varphi(x' - y + y_0) < 1$.

from $\varphi(y_0) = 1$. $\varphi(x') < \varphi(y)$.

Proof of Thm. $x \notin A \Rightarrow \exists$ wk neigh. of x (given by φ, C as above) separating x from A.