Haar measure (Folland §11.1)

a <u>locally compact group</u> G is

- group law (gh, g^{-1}, $e_G = gg^{-1}$) on G
- locally compact Hausdorff top. on G.

s.t. $G \times G \to G$, $(g, h) \to g^{-1}h$ is cont.

(then $(g, h) \mapsto gh$, $g \mapsto g^{-1}$ also)

Ex. matrix groups $GL_n(\mathbb{R})$, $O(n)$, $U(n)$, ...

$\prod_{i=1}^{\infty} \{0, 1\}$ either as $\varprojlim \mathbb{Z}/2^n\mathbb{Z}$

or $\prod_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$

A <u>left Haar measure</u> is a Radon measure

μ on G s.t. $\mu(gA) = \mu(A)$ $(g \in G, A \in \mathcal{B}_G)$

(same as positive func. $I_\mu : C_c(G) \to \mathbb{R}$,

$I_\mu(f) = I_\mu(L_g f)$ for $(L_g f)(h) = f(g^{-1}h)$)

Ex. $\mathbb{R}$! Th'm

Morally: look at the "covering number"

$(E : V) = \inf\{k : \exists g_1, \dots, g_k \in G \quad E \subset \bigcup_{i=1}^{k} g_i V\}$

for $E, V \subset G$

Fix $E_0 \in \mathcal{B}_G$, take neighborhoods $V_1 \supset V_2 \supset \dots$ of e_G.

$\rightsquigarrow$ normalized cov. num $M_k(E) = \dfrac{(E : V_k)}{(E_0 : V_k)}$

limit $\mu(E) = \lim_{k \to \infty} M_k(E)$

Ex. $G = \mathbb{R}$, $E_0 = [0, 1]$, $V_k = (-\frac{1}{2k}, \frac{1}{2k})$

$\Rightarrow (E, V_k) \sim k(b-a)$ for $E = (a, b)$ $a < b$.

so $M_k(E) \sim b - a = m(E)$.

Th'm. $\exists$ (nonzero) left Haar meas. on G

Analogue for funcs.

$f, F \in C_c(G)$ pos

$(f : F) = \inf \{ \sum_{i=1}^{n} c_i : \exists g_1, \dots, g_n$ s.t. $f \leq \sum c_i L_{g_i} F \}$

Fix $f_0 \in C_c(G)$ pos, put $I_F(f) = \frac{(f : F)}{(f_0 : F)}$

then $\frac{1}{(f_0 : f)} \leq I_F(f) \leq (f : f_0)$, $I_F(L_g f) = I_F(f)$

from $(f_1 : f_3) \leq (f_1 : f_2)(f_2 : f_3)$ $\quad$ $I_F(af) = a I_F(f)$ $(a \geq 0)$

Want to make sense of $I(f) = \lim_{F_i : \operatorname{supp}(F_i) \to \{e\}} I_{F_i}(f)$

use compactness to justify this.

$f \in C_c(G)$ pos $\to X_f = \left[\frac{1}{(f_0 : f)}, (f : f_0) \right]$.

$X = \prod_{\text{all } f \text{ as above}} X_f$ cpt. sp $\quad$ F defs a point

$p_F = (I_F(f))_f \in X$.

$V \subset G$ neigh. of e_G $\Rightarrow$ $K(V) =$ closure of $\{ p_F : \operatorname{supp}(F) \subset V \}$

- Claim. $\bigcap_{V \text{: as above}} K(V)$ is non-empty.
  - finite intersection is non-empty. $\bigcap_{i=1}^{n} K(V_i) \supset K(\bigcap_{i=1}^{n} V_i)$
  - nonempty intersection from cpt-ness of X.
- Claim $f_1, f_2 \in C_c(G)$ pos. $\Rightarrow \exists V$ (small) s.t.

(L.11.7) $\quad I_F(f_1) + I_F(f_2) \sim I_F(f_1 + f_2)$ $\quad$ ($\operatorname{supp}(F) \subset V$.)

$\rightsquigarrow$ $I \in \bigcap_{V \text{ as above}} K(V)$ represents a (invariant) lin. func.