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Problem Set 4

Prob. 7 (1) For any $A \in \mathcal{M}$, the series $\sum_{n=1}^{\infty} \nu_n(A)$ is absolutely convergent ; $|\nu_n(A)| \le |\nu_n|(A) \le |\nu_n|(X)$ and $\sum_{n=1}^{\infty} |\nu_n|(X) < \infty$ implies $\sum_n |\nu_n(A)| < \infty$

- $\nu(\phi) = 0$ from $\nu_n(\phi) = 0$
- $\nu(A) \in \mathbb{C}$ for $A \in \mathcal{M}$ from the above claim
- $\nu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \nu(A_i)$ for $A_1, A_2, \cdots \in \mathcal{M}$, mutually disjoint ; we have $\nu_n(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \nu_n(A_i)$

by assumption, so we want to compare $\sum_{n=1}^{\infty}\sum_{i=1}^{\infty} \nu_n(A_i)$ and $\sum_{i=1}^{\infty}\sum_{n=1}^{\infty} \nu_n(A_i)$. These are equal if absolute summability holds. We have

$$\sum_{i=1}^{\infty} |\nu_n(A_i)| \underset{\text{above estimate}}{\le} \sum_{i=1}^{\infty} |\nu_n|(A_i) \underset{|\nu_n| \text{ is a measure}}{=} |\nu_n|(\bigcup_{i=1}^{\infty} A_i) \le |\nu_n|(X)$$

So $\sum_n \sum_i |\nu_n(A_i)| \le \sum_n |\nu_n|(X) < \infty$.

(2) For any $A \in \mathcal{M}$, $|\nu_n(A)| \le \mu(A)$ implies $|\nu(A)| \le \mu(A)$ Given $A_1, A_2, \cdots \in \mathcal{M}$, mutually disjoint we have

$$\nu_n(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \nu_n(A_k)$$

The right hand side is absolutely summable and the k-th term is dominated by $\mu(A_k)$ Taking limit for $n \to \infty$ gives

$$\nu(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \nu(A_k).$$

with the same bound on k-th term.

$\nu(\phi) = \lim \nu_n(\phi) = 0$, $\nu(A) \in \mathbb{C}$ are obvious from assumptions

Problem 8 Given $E \in M_1 \otimes M_2$, we have

$$(\nu_1 \otimes \nu_2)(E) = \int d\nu_1(x_1) \int d\nu_2(x_2)\, 1_E(x_1, x_2)$$

$$= \int \nu_2(E_{x_1})\, d\nu_1(x_1)$$

for $E_{x_1} = \{x_2 \in X_2 : (x_1, x_2) \in E\}$

We have an analogous presentation for $\mu_1 \otimes \mu_2(E)$

(1) Suppose $(\mu_1 \otimes \mu_2)(E) = 0$. Then $\mu_2(E_{x_1}) = 0$ for μ_1-a.e. x_1. That is, $A = \{x_1 : \mu_2(E_{x_1}) \neq 0\}$ satisfies $\mu_1(A) = 0$.

Put $B = \{x_1 : \nu_2(E_{x_1}) \neq 0\}$. Then $\nu_2 \ll \mu_2$ implies $B \subset A$ so $\mu_1(B) = 0 \Rightarrow \nu_1(B) = 0$ and we get $\nu_2(E_{x_1}) = 0$ for ν_1-a.e. x_1.

This implies $\int \nu_2(E_{x_1})\, d\nu_1(x_1) = 0$.

(2) We want to show $(\nu_1 \otimes \nu_2)(E) = \int_E \frac{d\nu_1}{d\mu_1}(x_1) \frac{d\nu_2}{d\mu_2}(x_2)\, d(\mu_1 \otimes \mu_2)(x_1, x_2)$

We have $\nu_2(E_{x_1}) = \int_{E_{x_1}} \frac{d\nu_2}{d\mu_2}(x_2)\, d\mu_2(x_2)$

and $\int f(x_1)\, d\nu_1(x_1) = \int f(x_1) \frac{d\nu_1}{d\mu_1}(x_1)\, d\mu_1(x_1)$

for nonnegative measurable f, so

$$\int \nu_2(E_{x_1})\, d\nu_1(x_1) = \int d\mu_1(x_1) \int d\mu_2(x_2) \frac{d\nu_1}{d\mu_1}(x_1) \frac{d\nu_2}{d\mu_2}(x_2)\, 1_E(x_1, x_2)$$

$$= \int_E \frac{d\nu_1}{d\mu_1}(x_1) \frac{d\nu_2}{d\mu_2}(x_2)\, d(\mu_1 \otimes \mu_2)(x_1, x_2)$$

Problem 9

(1) $m << \mu$: $\mu(A) = 0$ only happens for $A = \emptyset$

so $\mu(A) = 0 \Rightarrow m(A) = 0$.

There is no p s.t. $\int f \, dm = \int f p \, d\mu$.

If there was such p, then $p(x) = 0$ for any $x \in [0, 1]$: $0 = m(\{x\}) = \int 1_{\{x\}}(t) \, dm(t)$

$= \int 1_{\{x\}} \cdot p \, d\mu = p(x)$.

But this implies $\int f p \, d\mu = 0$ for all f.

We cannot apply the Radon-Nikodym theorem because μ is not σ-finite

(2) the Lebesgue decomposition would be

$\mu = \mu_a + \mu_s$, $\mu_a << m$, $\mu_s \perp m$.

with $S \subset [0, 1]$ s.t. $\mu_s(A) = \mu(A \cap S)$,

$m(S) = 0$, $\mu_a(A) = \mu(A \setminus S)$,

Claim $S = [0, 1]$; otherwise take $x \notin S$

$\mu_a(\{x\}) = \mu(\{x\}) = 1$ but $m(\{x\}) = 0$

this contradicts $\mu_a << m$.

But $m(S) = 1$, this contradicts $m \perp \mu_s$.