Problem 1

a. Let μ be the Lebesgue measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, $\mu \otimes \mu$ be the product measure on $(\mathbb{R}^2, \mathcal{B}_{\mathbb{R}^2})$ for the σ-algebra $\mathcal{B}_{\mathbb{R}}$ of Borel subsets of $\mathbb{R}$.

Then we make sense of the area of C as $\mu \otimes \mu (C)$. By definition C is a countable union of closed sets of the form $[a,b] \times [c,d]$

Hence C is in $\mathcal{B}_{\mathbb{R}^2} = \mathcal{B}_{\mathbb{R}} \times \mathcal{B}_{\mathbb{R}}$

In general $\mu \otimes \mu$ has the following properties

- $\mu \otimes \mu ([a,b] \times [c,d]) = (b-a)(d-c)$, i.e. it agrees with the conventional notion of area for rectangles
- $\mu \otimes \mu (\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu \otimes \mu (A_i)$ for disjoint measurable sets $A_1, A_2, \cdots$; so it is consistent with partition into countably many measurable subsets (or ~~approximation by exhaustion by countably ---~~)

Hence it is reasonable to interpret $(\mu \otimes \mu)(B)$ as the area of $B \in \mathcal{B}_{\mathbb{R}^2}$

b. $A_n^{(m)}$ $(m = 1, 2, \cdots, \; 1 \le n \le 8^{m-1})$ are mutually disjoint, $\mu \otimes \mu (A_n^{(m-1)}) = \frac{1}{3^m} \times \frac{1}{3^m} = 3^{-2m}$

So $(\mu \otimes \mu)(C) = \sum_m \sum_{n=1}^{8^{m-1}} 3^{-2m}$.

cont, $= \sum_{m=1}^{\infty} \frac{1}{8} \cdot \left(\frac{8}{9}\right)^m = \frac{1}{8} \frac{\frac{8}{9}}{1-\frac{8}{9}} = 1$

<u>The area of C is 1</u> //

Problem 2

a. As measure spaces, we consider

$([0,\infty), \mathcal{B}_{[0,\infty)}, m)$ for the Lebesgue measure m.

$(\mathbb{N}, \mathfrak{P}(\mathbb{N}) = 2^{\mathbb{N}}, \mu)$ for the counting measure μ on $\mathbb{N}$

and the product measure space

$([0,\infty)\times\mathbb{N}, \mathcal{B}_{[0,\infty)} \otimes \mathfrak{P}(\mathbb{N}), m \otimes \mu)$

We take the function $f(x,n) = (-1)^n e^{-n^2 x}$

on $[0,\infty)\times\mathbb{N}$.

- This is measurable: it can be written as $f(x,n) = \sum_{k=1}^{\infty} 1_{\{k\}}(n)\,(-1)^n e^{-n^2 x}$;

$$1_{\{k\}}(n)\,(-1)^n e^{-n^2 x} = \begin{cases} (-1)^n e^{-n^2 x} & (n=k) \\ 0 & \end{cases}$$

this corresponds to a measurable function on $[0,\infty) \simeq \{k\}\times[0,\infty)$, and countable sum preserves measurability.

- This is integrable:

$$\int |f| \, d\, m\otimes\mu = \int d\mu \int dm \, |f| \quad \text{(Fubini)}$$

$$= \int_{\mathbb{N}} d\mu(n) \int_{[0,\infty)} dm(x)\, e^{-n^2 x}$$

cont. $= \sum_{n=1}^{\infty} \int_0^{\infty} e^{-n^2 x}\, dx = \sum_n \frac{1}{n^2} < \infty$

By Fubini's theorem again

$$\int d\mu \int dm\, f = \int f\, dm\otimes\mu = \int dm \int d\mu\, f$$

This means $\sum_n \int_0^{\infty} f(x,n)\, dx = \int_0^{\infty} \sum_n f(x,n)\, dx$

b. We do not have $\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$

$i = 1$: $\sum_j a_{1j} = 1 + 1 + 0 + \cdots = 2$

otherwise : $\sum_j a_{ij} = 0 + \cdots + \underset{j=i-1}{(-1)} + 0 + \underset{j=i+1}{1} + \cdots = 0.$

so $\sum_i \sum_j a_{ij} = 2$

$j = 1$: $\sum_i a_{i1} = 1 + (-1) + 0 + \cdots = 0$

otherwise : $\sum_i a_{ij} = 0 + \cdots + \underset{i=j-1}{1} + 0 + \underset{i=j+1}{(-1)} + \cdots = 0$

so $\sum_j \sum_i a_{ij} = 0$

If we define a measurable function f on $\mathbb{N}\times\mathbb{N}$ by $f(i,j) = a_{i,j}$, this is not integrable for $\mu\otimes\mu$:

$$\int |f|\, d\mu\otimes\mu = \sum_i \sum_j |a_{ij}| = \sum_i 2 = \infty$$

So Fubini's theorem does not apply

Problem 3

a $\|\phi_n\| = 1$

$\|\phi_n\| \le 1$ from $\left|\int f(x) g_n(x)\, dx\right| \le \int |f(x)|\, g_n(x)\, dx$

$\le \|f\|_\infty \int g_n\, dx. = \|f\|_\infty$

$\|\phi_n\| \ge 1$ by taking $f \in C_c(\mathbb{R})$ s.t. $f(x) = 1$ for $0 \le x \le \frac{1}{n}$, $|f(x)| \le 1$ for $x \in \mathbb{R}$

cont. then $|\phi_n(f)| = \int \underbrace{f\cdot g_n}_{"g_n} dx = 1.$

b. $\phi(f) = f(0)$

Fix $\varepsilon > 0$. Then $\exists N$ s.t. $|f(0) - f(x)| < \varepsilon$

for $0 < x < \frac{1}{N}$ by the continuity of f at 0

We claim that $|f(0) - \phi_n(f)| < \varepsilon$ for $n > N$.

indeed: $|f(0) - \phi_n(f)| = |\int (f(0) - f(x))\, g_n(x)\, dx|$

$|\int_0^{\frac{1}{n}} |f(0) - f(x)|\, g_n(x)\, dx| \leq \varepsilon \int g_n\, dx = \varepsilon.$

from $|f(0) - f(x)| < \varepsilon$ for $0 < x < \frac{1}{n} < \frac{1}{N}$.

c 1 for nontrivial (around 0) function f,

there is no integrable function $h(x)$.

s.t. $|f \cdot g_n| \leq |h|$ for all n

so $\int f \cdot g_n\, dx \to \int (\lim f \cdot g_n)\, dx$ is not guaranteed.

2. $\phi_g(f) = \int f g\, dx \quad (f \in C_c(\mathbb{R}))$

makes sense for $g \in L^1(\mathbb{R})$ and

we have $\|\phi_g\| = \|g\|_1$

We have pointwise convergence $g_n \to 0$

but $\|g_n\|_1 = 1 \not\to 0$ so $\phi_n = \phi_{g_n}$

does not converge to 0.

Problem 4

a $$\mu_+(A) = \int_B 1_A(x)\, e^{-x} \sin x \, dm(x) + \sum_{k=1}^{\infty} \frac{1}{(2k)^2} \delta_{2k}(A)$$

$$B = \bigcup_{k \in \mathbb{Z}} (2k\pi, (2k+1)\pi)$$

$$\mu_-(A) = -\int_{B^c} 1_A(x)\, e^{-x} \sin x \, dm(x) + \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} \delta_{2k+1}(A)$$

Put $C = B \cup 2\mathbb{N} \setminus (2\mathbb{N}+1)$, $D = B^c \cup (2\mathbb{N}+1) \setminus 2\mathbb{N}$

Then $\mu_+(A) = \mu_+(A \cap C)$, $\mu_-(A) = \mu_-(A \cap D)$

$\mu_+(D) = 0$, $\mu_-(C) = 0$

so $\mu_+ \perp \mu_-$, $\mu = \mu_+ - \mu_-$

b. $$\mu_a^+(A) = \int_B 1_A(x)\, e^{-x} \sin x \, dm(x)$$

$$\mu_s^+(A) = \sum \frac{1}{(2k)^2} \delta_{2k}(A)$$

$\mu_s^+ \perp m$ follows from $\delta_{2k} \perp m$

$\mu_a^+ \ll m$ follows from integral procentation

$$m(A) = 0 \Rightarrow \mu_a^+(A) = \int_{B \cap A} e^{-x} \sin x \, dm(x) = 0$$

Problem 5

Let (Ω, F, μ) be a probability space,

$f_X : \Omega \to [0, \infty)$ measurable function

modelling X. So $\mathbb{E}[X] = \int f_X \, d\mu$.

Consider $E = \{ (\omega, x) \in \Omega \times [0, \infty),\ f(\omega) \geq x \}$

so $f(\omega) = \int_{[0,\infty)} 1_E(\omega, x) \, dm(x)$ for the Lebesgue measure m.

Cont: By Tonelli's theorem

$$\int d\mu \int dm \, 1_E = \int dm \int d\mu \, 1_E.$$

For $t \in [0, \infty)$ $\int 1_E(\omega, t) \, d\mu(\omega) = \mu(\{\omega : f(\omega) \geq t\})$

$$= \mathbb{P}[X \geq t].$$

We get $\mathbb{E}[X] = \int d\mu \int dm \, 1_E = \int \mathbb{P}[X \geq t] \, dt$

Problem 6

a. We need to check $\phi(\delta_n) \to 0$ $(n \to \infty)$

for all $\phi \in \ell^2(\mathbb{N})^*$

By the Riesz representation theorem

any ϕ is of the form $\xi \mapsto (\xi, \eta)$

for $\eta \in \ell^2(\mathbb{N})$

We have $(\delta_n, \eta) = \eta(n)$, and $\|\eta\|_2 < \infty$

implies $\eta(n) \to 0$ $(n \to \infty)$.

b We need to find $\phi \in \ell^1(\mathbb{N})^*$ s.t.

$\phi(\delta_n) \not\to 0$ $(n \to \infty)$.

We put $\phi(\xi) = \sum_{n=1}^{\infty} \xi(n)$ for $\xi \in \ell^1(\mathbb{N})$

Then $|\phi(\xi)| \leq \sum |\xi(n)| =: \|\xi\|_1$, hence

$\|\phi\| = 1$.

On the other hand $\phi(\delta_n) = 1$ for all n.

Problem 7

Let μ_0 be the probability measure on $\mathbb{T}$ characterized by $\int f \, d\mu_0 = \int_0^1 f(e^{2\pi i t}) \, dt$

By the Fourier transform $\ell^2 \mathbb{Z} \to L^2(\mathbb{T}, \mu_0)$

$\xi \mapsto \hat{\xi}$ we have $\langle \xi, \eta \rangle = \int \hat{\xi} \, \overline{\hat{\eta}} \, d\mu_0$

Claim: the measure μ on $\mathbb{T}$ s.t. $d\mu = |\hat{\eta}|^2 d\mu_0$ satisfies the condition of the problem.

Proof By linearity we may assume $p(t) = t^k$ for some $k \in \mathbb{Z}$ Then

$$\phi(p) = \langle U^k \eta, \eta \rangle = \sum_n \eta(n-k) \overline{\eta(n)}.$$

$$\int p(m) \, d\mu(m) = \int_0^1 e^{2\pi i k t} |\hat{\eta}(e^{2\pi i t})|^2 \, dt$$

This is the inverse Fourier transform of $|\hat{\eta}|^2$.

- The inverse Fourier transform of $\hat{\eta}$: η
- ~~~~ of $\overline{\hat{\eta}} = \overline{\eta(-n)}$
- ~~~~ of $f_1 \cdot f_2$ = convolution prod. of the inverse F-transforms of f_i

Thus $\int_0^1 e^{2\pi i k t} |\hat{\eta}(e^{2\pi i t})|^2 \, dt = (\eta * \overline{\eta(-\sim)})(-k)$

$$= \sum_n \eta(n-k) \overline{\eta(n)}.$$