How to "think" about problems

Prob 1.

overall goal: get bij. map f: from $X = \prod_{i=1}^{\infty} \{0, 1\}$ to $[0, 1]$ compat w/ measures.

important steps: 1. [use binary expansion of numbers $0 \le t \le 1$ to] get an approximate sol.

2. check compat w/ measure for easy sets

3. how gen case reduce to 2.

4. how to make f bijective

Not so important: precise form of res. in 4.

What is an short answer for each step?

1 →

2 $Y_1 \times \cdots \times Y_M \times \prod_{i=M+1}^{\infty} \{0, 1\}$ $\quad Y_i \subset \{0, 1\}$

will map to fin. union of intervals.

3 $f_* \mu = m$ holds on a generating collection of $\mathcal{B}[0, 1]$.

4 "bad points" ($x \in X$ s.t. $\exists y \neq x$ $f(x) = f(y)$).

are countable, countable sets are μ & m-null

More detailed answer

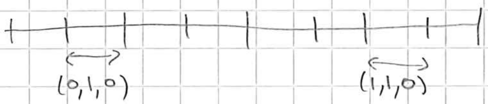
3 (intertwined with 2)

$(y_1, \cdots, y_M) \in Y_1 \times \cdots \times Y_M$ $\quad y_i = 0, 1$

contributes to the interval

start : $\frac{y_1}{2} + \frac{y_2}{4} + \cdots + \frac{y_M}{2^M}$

length : 2^{-M}.



$M=3$

and we get all intervals like this (say $\mathcal{D}$).

⇝ we get arbitrarily small pieces

what can we get? from ctble union, complement

$A \subset [0,1]$ ⇝ collect all intervals $I_1, I_2, \dots$
as above, s.t. $I_j \subset A$.

⇝ $\bigcup_{j=1}^{\infty} I_j \subset A$ equality if A open.

$\forall\, t \in A \quad \exists\, I$ as above $t \in I \subset A$.

so $\sigma(\mathcal{D})$ contains all open sets. $\Rightarrow \sigma(\mathcal{D}) = \mathcal{B}_{[0,1]}$

4. removing null sets doesn't change essential info.

$A \subset X$ "bad pts" is ctble.

$f(A) = \{ \frac{n}{2^M} \in [0,1],\ n \in \mathbb{N} \}$ also ctbl

$f|_{X \setminus A}$ is bijective to $[0,1] \setminus f(A)$

$f'\ A \to f'(A)$ bij. map

- possible bc. A & $f(A)$ both ctbl infin.
- don't care prec. form
  (because it's not needed)

More on mandatory assignments

Problem 3 : $(\Omega, \mathcal{F}, \mathbb{P})$ prob sp.

X, Y (indep) rand. vars.

- generally (w/o indep.)

  joint distribution is given by

  the prob. meas. on $\mathbb{R}^2$

  $\mu_{X,Y}(E) = \mathbb{P}[(X,Y) \in E)$ $\quad E \in \mathcal{B}_{\mathbb{R}^2}$

  also write $\mathbb{P}_{X,Y}$

  i.e. pushforw. $(f_{X,Y})_* \mathbb{P}$ for

  $f_{X,Y} : \Omega \to \mathbb{R}^2, \quad \omega \mapsto (X(\omega), Y(\omega))$

- prob. measure for X

  $\mu_X(A) = \mathbb{P}[X \in A] = (X_* \mathbb{P})(A) \quad A \in \mathcal{B}_{\mathbb{R}}$

  also write $\mathbb{P}_X$ $\qquad$ as $\Omega \to \mathbb{R}$

  representing the "distribution" of X

  (not to be confused w Cumulative

  Distribution Function $F_X(t) = \mathbb{P}[X \leq t]$)

- X & Y indep. means

  $\underline{\mathbb{P}[X \in A, Y \in B]} = \mathbb{P}[X \in A]\, \mathbb{P}[Y \in B]$

  is eq. to $\mathbb{P}[(X,Y) \in A \times B]$

  i.e. $\mu_{X,Y}(A \times B) = (\mu_X \otimes \mu_Y)(A \times B)$

  for $A, B \in \mathcal{B}_{\mathbb{R}}$.

  $\{A \times B : A, B \in \mathcal{B}_{\mathbb{R}}\}$ gen. $\mathcal{B}_{\mathbb{R}^2}$

  $\rightsquigarrow \mu_{X,Y} = \mu_X \otimes \mu_Y$

So we can model the joint distr. of X & Y

by $(\mathbb{R}^2, \mathcal{B}_{\mathbb{R}^2}, \mu_X \otimes \mu_Y)$