MAT4410 (2020 AUTUMN) EXERCISE SET 3

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Problem 5 (Mean value inequality for vector valued functions, Exercise 25.4)**.** Let X be a Banach space, $a < b$ be real numbers, and $f(x)$, $f'(x)$ be X-valued functions defined on $a \leq x \leq b$ such that

$$f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} \quad (a < t < b).$$

(That is, $f(x)$ is an X-valued C^1-class function on $a \leq x \leq b$.) We want to show

$$\|f(b) - f(a)\| \leq \int_a^b \|f'(t)\| \, dt (\leq (b-a) \sup_x \|f'(x)\|).$$

(1) Use the Hahn–Banach theorem to estimate $\|f(b) - f(a)\|$ in terms of $\phi(f(b))$ and $\phi(f(a))$ for $\phi \in X^*$ (with $\|\phi\| \leq 1$ if you want).
(2) Use (1) and the usual mean value inequality for scalar valued functions $\phi(f(x))$ to get the above inequality.

Problem 6 (Exercise 25.7)**.** Let X be a normed vector space, and $M \subset X$ be a finite dimensional subspace. We want to show that there is a closed subspace $N \subset X$ such that $X = M \oplus N$ (that is, M is a *complemented* subspace).

(1) When M is 1-dimensional, find N as the kernel of a suitable functional on X. (You want to use the Hahn–Banach theorem here.)
(2) How do you extend (1) to finite dimensional case in general?

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