MAT4410 (2020 AUTUMN) EXERCISE SET 4

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Problem 7 (Exercise 24.6)**.** Let $(X, \mathcal{M})$ be a measurable space, and $\nu_1, \nu_2, \dots$ be complex measures on $(X, \mathcal{M})$. We put

$$|\nu_n|(A) = \sup\Big\{\sum_{i=1}^{n} |\nu_n(A_i)| \;\Big|\; A = \bigcup_i A_i, A_i \cap A_j = \emptyset \quad (i \neq j)\Big\}$$

(this is the total variation of ν_n which appeared in September 22).

(1) Show that, if $\sum_n |\nu_n|(X) < \infty$, then $\nu(A) = \sum_n \nu_n(A)$ defines a complex measure.
(2) Suppose that there is a finite (positive) measure μ on $(X, \mathcal{M})$ such that $|\nu_n|(A) \leq \mu(A)$ holds for any n and $A \in \mathcal{M}$, and that $\lim_n \nu_n(A)$ exists. Show that $\nu(A) = \lim_n \nu_n(A)$ defines a complex measure on $(X, \mathcal{M})$.

Problem 8 (Exercise 24.7)**.** Let $(X_1, \mathcal{M}_1, \mu_1)$ and $(X_2, \mathcal{M}_2, \mu_2)$ be σ-finite measurable spaces. Suppose moreover that ν_1 is a σ-finite (positive) measure on $(X_1, \mathcal{M}_1)$ satisfying $\nu_1 \ll \mu_1$, and that ν_2 is a similar one for $(X_2, \mathcal{M}_2, \mu_2)$.

(1) Show $\nu_1 \otimes \nu_2 \ll \mu_1 \otimes \mu_2$.
(2) Show
$$\frac{d(\nu_1 \otimes \nu_2)}{d(\mu_1 \otimes \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1)\frac{d\nu_2}{d\mu_2}(x_2).$$

Problem 9 (Exercise 24.8)**.** We consider the measurable space $([0, 1], \mathcal{B}_{[0,1]})$. Let m be (the restriction of) the Lebesgue measure, and μ be the counting measure, both on $[0, 1]$.

(1) Show that $m \ll \mu$, but yet there is not function ρ on $[0, 1]$ satisfying $\int f dm = \int f \rho d\mu$. What goes wrong if you naively try to apply the Radon–Nikodym theorem?
(2) Show that μ has no Lebsuge decomposition relative to m.

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