MAT4410 (2020 AUTUMN) MOCK EXAM

MAKOTO YAMASHITA

Problem 1. (1) Give a measure-theoretic justification for the following argument. Suppose that $D \subset \mathbb{R}^2$ has a decomposition $D = \bigcup_{i=1}^{\infty} [a_i, b_i) \times [c_i, d_i)$ for some numbers a_i, b_i, c_i, d_i $(i = 1, 2, \dots)$. Then the area of D is given by $\sum_{i=1}^{\infty} (b_i - a_i) \times (d_i - c_i)$.

(2) Give a concrete computation of the area of $D = \{(x, y) \mid 0 \leq x, y, x + y < 1\} \subset \mathbb{R}^2$ following the above argument.

(3) What is wrong with the following argument?: Given $D \subset \mathbb{R}^2$, we put $D_x = \{y \in \mathbb{R} \mid (x, y) \in D\}$. Then $D^x = \{x\} \times D_x$ is a subset of $\{x\} \times \mathbb{R}$, but the latter is a straight line in $\mathbb{R}$ and should have the area 0. Then D^x also has the area 0. Since $D = \bigcup_{x \in \mathbb{R}} D^x$, the area of D is just the sum of 0's, which is 0.

Problem 2. (1) Explain the measure-theoretic framework for manipulations of the form

$$\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f_n(x) dx = \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} f_n(x) dx.$$

(Give the relevant measure spaces also.)

(2) Explain when you can justify this manipulation.

Problem 3. Consider the sequence of real functions given by

$$f_n(x) = \begin{cases} n|x| & (|x| \leq \frac{1}{n}) \\ 1 & (\frac{1}{n} < |x| \leq 1) \\ 2 - |x| & (1 < |x| \leq 2) \\ 0 & (2 < |x|). \end{cases}$$

(1) Find the limit $f \in C_c(\mathbb{R})$ with respect to the 2-norm topology.

(2) We do not have $\lim_n f_n(0) = f(0)$, but why is it compatible with the above convergence in the 2-norm topology?

Problem 4. Consider the signed measure μ on $\mathbb{R}$ given by

$$\mu(A) = \int_A e^{-x^2} \sin x dx + \delta_0(A).$$

(1) What is the Jordan decomposition for μ?

(2) What is the Lebesgue decomposition relative to the Lebesgue measure on $\mathbb{R}$ give for the result of above?

To score better, answer one of the following problems.

Problem 5. Compute the area of $D \subset [0, 1]^2$ given by the following procedure.

(1) Set $D_0 = [0, 1]^2$.

(2) Divide D_0 into 9 equal small squares, and remove the middle one. Call the remaining part D_1.

(3) Divide each small square (from the previous step) in D_1 again into 9 small squares, and remove the middle one. Call the union of the remaining squares D_2.

(4) Repeat this process and inductively define $D_3 \supset D_4 \supset \dots$.

(5) Set $D = \bigcap_n D_n$.

Problem 6. Recall that the characteristic function ϕ_X of a random variable X is given by $\phi_X(t) = \mathbb{E}[e^{itX}]$. Given independent random variables X and Y, show $\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t)$.

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