1. Let (X, d) be a metric space. Show that the space of bounded Borel functions on X is the smallest space of functions that contains the space of bounded continuous functions with bounded support (that is, contained in a ball of finite radius) and is closed under pointwise limits of bounded sequences.

2. Let ν and μ be measures on a measurable space $(X, \mathcal{B})$, and $\nu \ll \mu$. We know that if $\nu(X) < \infty$, then $\forall \varepsilon > 0\ \exists \delta > 0$ such that if $\mu(A) < \delta$, then $\nu(A) < \varepsilon$. Give an example showing that without the assumption $\nu(X) < \infty$ this is not always true.

3. Recall that a function $f\colon X \to [-\infty, +\infty]$ on a topological space X is called lower (resp., upper) semicontinuous if

$$\liminf_{x \to x_0} f(x) \ge f(x_0) \quad (\text{resp.,} \quad \limsup_{x \to x_0} f(x) \le f(x_0))$$

for all $x_0 \in X$.

(i) Show that lower (resp., upper) semicontinuity is equivalent to requiring $f^{-1}((a, +\infty])$ (resp., $f^{-1}([-\infty, a))$) to be open for all $a \in \mathbb{R}$. Therefore the semicontinuous functions are Borel.

(ii) Show that if $(f_i)_{i \in I}$ is a collection of lower (resp., upper) semicontinuous functions, then the function f defined by

$$f(x) = \sup_{i \in I} f_i(x) \quad (\text{resp.,} \quad f(x) = \inf_{i \in I} f_i(x))$$

is lower (resp., upper) semicontinuous.

4. Let μ be a Radon measure on $\mathbb{R}^n$ and λ_n the Lebesgue measure.

(i) Show that, for all $r > 0$, the function $x \mapsto \mu(B_r(x))$ is lower semicontinuous, while $x \mapsto \mu(\overline{B_r(x)})$ is upper semicontinuous. Do we have continuity?

(ii) Define $\underline{D}\mu(x) = \liminf_{r \downarrow 0} \dfrac{\mu(B_r(x))}{\lambda_n(B_r(x))}$. Show that

$$\underline{D}\mu(x) = \liminf_{r \downarrow 0} \frac{\mu(\overline{B_r(x)})}{\lambda_n(B_r(x))} = \lim_{n \to \infty} \inf_{0 < r < \frac{1}{n}} \frac{\mu(\overline{B_r(x)})}{\lambda_n(B_r(x))}.$$

Conclude that the function $\underline{D}\mu$ is Borel.

5. Let μ be a Radon measure on $\mathbb{R}^n$.

(i) Using Vitali's covering lemma and arguments similar to the proof of the maximal inequality, show that if $A \subset \mathbb{R}^n$ is a Borel set such that $\underline{D}\mu(x) < \alpha$ for every $x \in A$, then

$$\mu(A) \le 5^n \alpha \lambda_n(A).$$

(In fact, the factor 5^n is superfluous here, but Vitali's covering lemma is not enough to show this, one has to use the Besicovitch covering theorem.)

(ii) Consider the Lebesgue decomposition $\mu = \mu_a + \mu_s$ of μ with respect to λ_n. Arguing as in the proof of the Lebesgue differentiation theorem show that $\underline{D}\mu(x) = +\infty$ for μ_s-a.e. x. Conclude that if μ is differentiable at every point, then $\mu \ll \lambda_n$.