1. Let λ_n be the Lebesgue measure on $\mathbb{R}^n$.

(i) Assume A and B are Borel sets of nonzero measure. Show that there are a Borel subset $C \subset A$ of nonzero measure and $x \in \mathbb{R}^n$ such that $x + C \subset B$.
Hint: if this is not true, then $\int \chi_A(y-x)\chi_B(y)d\lambda_n(y) = 0$ for all x.

(ii) With A and B as above, assume also that $\lambda_n(A) = \lambda_n(B) < \infty$. Show using Zorn's lemma that there exist disjoint Borel subsets $C_k \subset A$ ($k \in \mathbb{N}$) and points $x_k \in \mathbb{R}^n$ such that $\lambda(A \setminus \cup_{k=1}^{\infty} C_k) = 0$, the sets $x_k + C_k$ are disjoint, contained in B, and

$$\lambda_n\big(B \setminus \cup_{k=1}^{\infty}(x_k + C_k)\big) = 0.$$

In other words, modulo sets of measure zero, we can cut A into countably many Borel pieces and use only translations in $\mathbb{R}^n$ to assemble B out of them.

2. Assume $f\colon \mathbb{R}^n \to \mathbb{C}$ is an integrable function.

(i) Consider the functions

$$\phi_k(x) = k^n e^{-\pi k^2 |x|^2}.$$

We already know that $\phi_k * f \to f$ in $L^1(\mathbb{R}^n)$. Show that for every point of continuity x of f we have $(\phi_k * f)(x) \to f(x)$.

(ii) Show that if f is continuous at 0 and $\hat{f} \geq 0$, then $\hat{f}$ is integrable and

$$\int_{\mathbb{R}^n} \hat{f}(\xi) d\xi = f(0).$$

Hint: recall the proof of the Fourier inversion formula.

3. A (unitary) *character* on a locally compact abelian group G is a continuous homomorphism $\chi\colon G \to \mathbb{T}$.

(i) Show that every character of the additive group $\mathbb{R}$ has the form $\chi(x) = e^{ixy}$ for a uniquely defined $y \in \mathbb{R}$.
Hint: show first that χ is smooth by considering $\phi * \chi$ for a suitable $\phi \in C_c^{\infty}(\mathbb{R})$.

(ii) Show that every character of the multiplicative group $\mathbb{T}$ has the form $\chi(z) = z^n$ for a uniquely defined $n \in \mathbb{Z}$.