1. Assume X is a normed vector space and $A \subset X$ is a convex subset with nonempty interior. Show that the interior of A is convex and dense in A.

2. Consider a finite dimensional normed vector space X. Assume $A \subset X$ is a convex subset such that $0 \in A$ and the linear span of A is the entire space X. Show that the interior of A is not empty.

Applications of the Hahn-Banach theorem

3. Let $p_1, \dots, p_n$ be seminorms on a vector space X. Assume f is a linear functional on X such that $|f(x)| \le \sum_{k=1}^n p_k(x)$ for all $x \in X$. Show that there exist linear functionals f_k such that $f = \sum_{k=1}^n f_k$ and $|f_k(x)| \le p_k(x)$ for all $x \in X$ and $k = 1, \dots, n$.

Hint: consider the space X^n and its subspace consisting of the vectors $(x, \dots, x)$, $x \in X$.

4. Let X be an infinite dimensional normed space.

(i) Construct by induction vectors $x_n \in X$ and linear functionals $f_n \in X^*$ such that $\|x_n\| = \|f_n\| = f_n(x_n) = 1$ for all n and $f_n(x_m) = 0$ for all $n < m$.

(ii) Show that if X is complete, then there exists an injective linear map $\ell^\infty \to X$.

(iii) Show that the vector space ℓ^∞ has a continuum of linearly independent vectors.

Conclude that any infinite dimensional Banach space has at least a continuum of linearly independent vectors.

Applications of the Baire category theorem

5. Show that the subset $D \subset C[a, b]$ of functions differentiable at least at one point is meager (in the supremum norm), that is, it is a countable union of nowhere dense sets. Conclude that the continuous functions that are not differentiable at any point are dense in $C[a, b]$.

Hint: observe that $D \subset \cup_n D_n$, where

$$D_n = \{f : |f(x) - f(x_0)| \le n|x - x_0| \text{ for some } x_0 \text{ and all } x\}.$$

6. Let X and Y be complete metric spaces and $f\colon X \times Y \to \mathbb{C}$ be a function that is continuous in each variable. Show that there exists a point of continuity of f. Conclude that the set of points of continuity of f is dense in $X \times Y$.

Hint: for $\varepsilon > 0$, consider the sets $A_n \subset X \times Y$ consisting of points (x_0, y_0) such that $|f(x_0, y_0) - f(x, y_0)| < \varepsilon$ and $|f(x_0, y_0) - f(x_0, y)| < \varepsilon$ whenever $d(x, x_0) < 1/n$ and $d(y, y_0) < 1/n$, and observe that $\cup_n A_n = X \times Y$.