Applications of the open mapping theorem and the uniform boundedness principle

1. Let X be a Banach space. A subspace $Y \subset X$ is called *complemented* if there exists a closed subspace $Z \subset X$ such that X is the direct sum of Y and Z in the usual linear algebraic sense (in other words, $Y + Z = X$ and $Y \cap Z = 0$). Then Z is called complementary to Y.

(i) Show that if Y and Z are complementary closed subspaces, then there is $C > 0$ such

$$\|y\| + \|z\| \le C\|y + z\|$$

for all $y \in Y$ and $z \in Z$. It follows that the projection $P\colon X \to Y$ of X onto Y along Z (that is, $P(y + z) = y$) is a bounded linear operator.

(ii) Show that conversely, if there is a bounded projection $P\colon X \to Y$ (that is, P is a bounded surjective linear operator and $Py = y$ for $y \in Y$), then $\ker P$ and Y are complementary to each other.

2. Assume X and Y are Banach spaces and $T\colon X \to Y$ is a linear operator such that $f \circ T \in X^*$ for every $f \in Y^*$. Show that T is bounded.

3. Assume X and Y are Banach spaces and $\{T_n\}_n$ is a sequence of bounded linear operators $X \to Y$ such that the limit $Tx = \lim_n T_n x \in Y$ exists for every $x \in X$. Show that $T\colon X \to Y$ is a bounded linear operator.

4. Show by induction on n that if $B\colon X_1 \times \dots \times X_n \to Y$ is a map of Banach spaces that is continuous and linear in each variable, then there exists $C > 0$ such that

$$\|B(x_1, \dots, x_n)\| \le C\|x_1\| \dots \|x_n\|$$

for all $x_1 \in X_1$,..., $x_n \in X_n$. (The case $n = 2$ was done in the class.)

Measure theory

5. Consider the completion $(X, \bar{\mathcal{B}}, \bar\mu)$ of a measure space $(X, \mathcal{B}, \mu)$. Show that for any $\bar{\mathcal{B}}$-measurable function f on X, there exists a $\mathcal{B}$-measurable function $\tilde f$ such that $f = \tilde f$ $\bar\mu$-a.e. Hint: consider first $f \ge 0$.

6. Assume $\mathcal{B}$ is an algebra of subsets of X and μ is a finite premeasure on $(X, \mathcal{B})$. Let μ^* be the corresponding outer measure on X. Recall that a subset $A \subset X$ is called Caratheodory measurable if

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \setminus A) \ \text{ for all subsets } B \subset X.$$

Denote by $\mathcal{C}$ the collection of all Caratheodory measurable sets and recall that Caratheodory's theorem states that $\mathcal{C}$ is a σ-algebra containing $\mathcal{B}$ and $\mu^*|_{\mathcal{C}}$ is a measure extending μ.

(i) Show that $(X, \mathcal{C}, \mu^*|_{\mathcal{C}})$ coincides with the completion of $(X, \sigma(\mathcal{B}), \mu^*|_{\sigma(\mathcal{B})})$, where $\sigma(\mathcal{B})$ is the σ-algebra generated by $\mathcal{B}$. Therefore $\mathcal{C}$ is generated as a σ-algebra by $\mathcal{B}$ and the sets of outer measure zero.

(ii) Show that a subset $A \subset X$ lies in $\mathcal{C} = \overline{\sigma(\mathcal{B})}$ if and only if for every $\varepsilon > 0$ there exists $B \in \mathcal{B}$ such that $\mu^*(A \Delta B) < \varepsilon$.

(iii) Show independently of (ii) and Caratheodory's theorem that if ν is a measure on $\sigma(\mathcal{B})$ extending μ, then for every $A \in \sigma(\mathcal{B})$ and $\varepsilon > 0$ there exists $B \in \mathcal{B}$ satisfying $\nu(A \Delta B) < \varepsilon$.

7. Let $(X_1, \mathcal{B}_1, \mu_1)$ and $(X_2, \mathcal{B}_2, \mu_2)$ be σ-finite measure spaces.

(i) Show using parts (ii) or (iii) from the previous exercise that for every set $A \in \mathcal{B}_1 \times \mathcal{B}_2$ such that $(\mu_1 \times \mu_2)(A) < \infty$ and every $\varepsilon > 0$ there exists a set $B \subset X_1 \times X_2$ that is a finite disjoint union of sets of the form $B_1 \times B_2$, with $B_i \in \mathcal{B}_i$ and $\mu_i(B_i) < \infty$, such that $(\mu_1 \times \mu_2)(A \Delta B) < \varepsilon$.

(ii) Take $1 \le p < \infty$. Show that the subspace of $L^p(X_1 \times X_2, \mathcal{B}_1 \times \mathcal{B}_2, d(\mu_1 \times \mu_2))$ spanned by functions of the form $(x_1, x_2) \mapsto f_1(x_1) f_2(x_2)$, with $f_i \in L^p(X_i, \mathcal{B}_i, d\mu_i)$, is dense.

Warning: this is not true for $p = \infty$.

8. Let $(X_1, \mathcal{B}_1, \mu_1)$ and $(X_2, \mathcal{B}_2, \mu_2)$ be σ-finite complete measure spaces. Consider the measure $\mu_1 \times \mu_2$ and denote by μ its completion.

(i) Show that if $A \subset X_1 \times X_2$ is a set of μ-measure zero, then for μ_1-a.e. x the set $A(x_1) = \{x_2 \mid (x_1, x_2) \in A\}$ has μ_2-measure zero.

(ii) Formulate and prove analogues of the Tonelli and Fubini theorems for the measure μ. Hint: use (i) and Exercise 5.