Finite Borel measures ~~and~~ as Lebesgue-Stieltjes measures

We consider $\Omega = \mathbb{R}$ and $\mathcal{B}$ = the σ-algebra of all Borel subsets of $\mathbb{R}$.

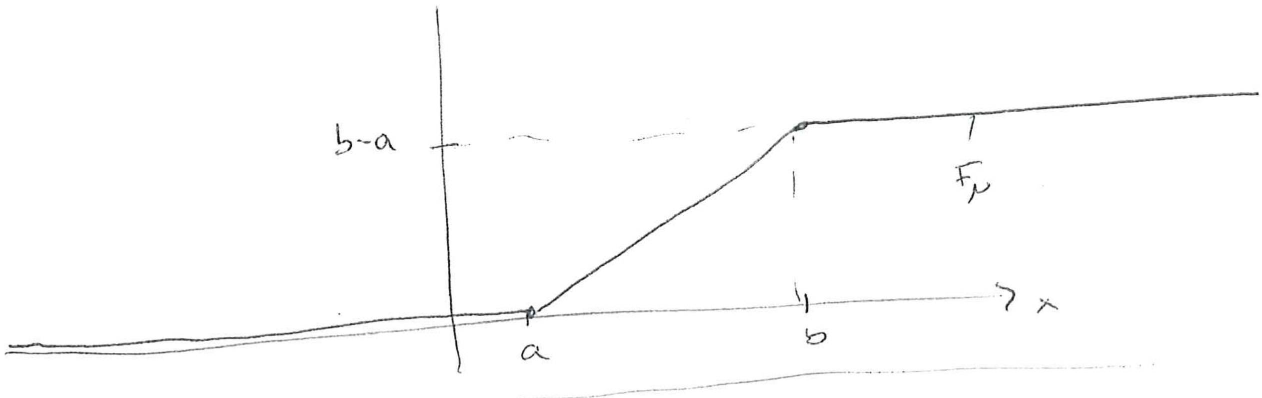
A measure μ on $(\mathbb{R}, \mathcal{B})$ is called a Borel measure (on $\mathbb{R}$). It is finite if $\mu(\mathbb{R}) < \infty$.

Def. Let μ be a finite Borel measure (on $\mathbb{R}$). The function $F_\mu : \mathbb{R} \to [0, \infty)$ given by $F_\mu(x) = \mu((-\infty, x])$ for $x \in \mathbb{R}$ is called the (cumulative) distribution function of μ.

Ex 1 $a < b$, $\mu(E) = \lambda(E \cap [a,b])$, $E \in \mathcal{B}$. (i.e. $\mu = \lambda_{[a,b]}$)

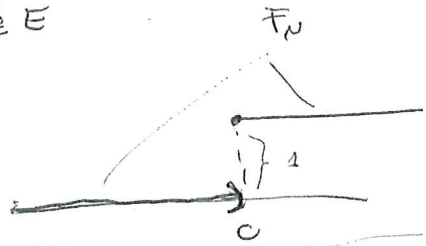
$\uparrow$ leb. meas.

Then
$$F_\mu(x) = \lambda((-\infty, x] \cap [a,b]) = \begin{cases} \lambda(\emptyset) = 0 & \text{if } x < a \\ \lambda(\{a\}) = 0 & \text{if } x = a \\ \lambda([a,x]) = x - a & \text{if } a < x \leq b \\ \lambda([a,b]) = b - a & \text{if } b < x \end{cases}$$



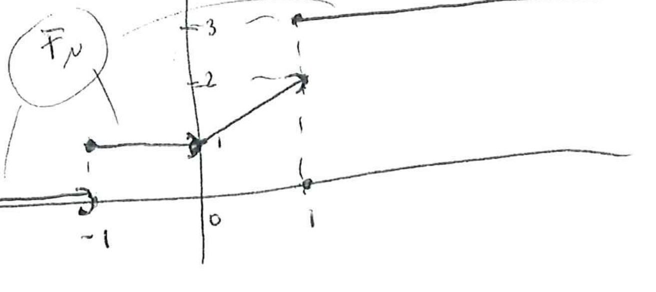
Ex 2 $c \in \mathbb{R}$, $\mu = \delta_c$, i.e. $\mu(E) = \begin{cases} 1 & \text{if } c \in E \\ 0 & \text{if } c \notin E \end{cases}$

Then $F_\mu(x) = \delta_c((-\infty, x]) = \begin{cases} 1 & \text{if } c \leq x \\ 0 & \text{if } x < c \end{cases}$



Ex 3 $\mu = \lambda_{[0,1]} + \delta_{(-1)} + \delta_1$

$F_\mu = F_{\lambda_{[0,1]}} + F_{\delta_{(-1)}} + F_{\delta_1}$



As we can see in these examples, the following holds:

$\boxed{F_\mu \text{ is } \overset{\text{monotone}}{\text{non-decreasing}}, \text{ bounded, right-continuous (on } \mathbb{R}\text{), and } \lim_{x \to -\infty} F_\mu(x) = 0}$

We recall that $F: \mathbb{R} \to \mathbb{R}$ is <u>right-cont. at $a \in \mathbb{R}$</u> if

$$F(a) = \underbrace{\lim_{x \to a^+} F(x)}_{\text{often denoted by } F(a^+)}$$

Note: $F(a^+)$ always exists when F is non-decreasing and is then given by

$$F(a^+) = \inf \{ F(x),\ a < x \} = \lim_{n \to \infty} F(x_n)$$

for every $\{x_n\} \subseteq (a, \infty)$ s.t. $x_n \to a$.

The first two properties of F_μ are obvious.

Let us check that <u>F_μ is right-cont. at $a \in \mathbb{R}$</u>:

$$F_\mu(a^+) = \lim_{n \to \infty} F_\mu\left(a + \tfrac{1}{n}\right)$$

$$= \lim_{n \to \infty} \mu\big(\underbrace{(-\infty, a + \tfrac{1}{n}]}_{J_n}\big)$$

$$= \mu\Big(\bigcap_{n=1}^{\infty} J_n\Big) = \mu((-\infty, a]) = F_\mu(a).$$

$\boxed{\begin{array}{l} \mu(J_1) < \infty \\ J_{n+1} \subseteq J_n \ \forall n \end{array}}$

Similarly, since F_μ is $\overset{\text{monotone}}{\text{non-decreasing}}$, we get that

$$\lim_{x \to -\infty} F_\mu(x) = \lim_{n \to +\infty} F_\mu(-n) = \lim_{n \to \infty} \mu((-\infty, -n]) = \mu\Big(\bigcap_{n=1}^{\infty} (-\infty, -n]\Big)$$

$$= \mu(\emptyset) = 0.$$

We also note that $\lim_{x \to \infty} F_\mu(x)$ exists and is equal to $\mu(\mathbb{R})$.

We therefore set $\boxed{F_\mu(-\infty) := 0,\ F_\mu(\infty) = \mu(\mathbb{R})}$.

Recall that the family $\mathcal{J} = \{(a,b] : -\infty \le a \le b < \infty\} \cup \{(c,\infty) : -\infty \le c < \infty\}$

is a semialgebra of subsets of $\mathbb{R}$ such that $\sigma(\mathcal{J}) = \mathcal{B}$.

Note that we can recover μ on $\mathcal{J}$ from F_μ:

- if $-\infty \le a \le b < \infty$, then
$$\mu((a,b]) = \mu((-\infty,b] \setminus (-\infty,a]) = \mu((-\infty,b]) - \mu((-\infty,a]) = F_\mu(b) - F_\mu(a)$$

- if $-\infty \le c < \infty$, then
$$\mu((c,\infty)) = \mu(\mathbb{R} \setminus (-\infty,c]) = \mu(\mathbb{R}) - \mu((-\infty,c]) = F_\mu(\infty) - F_\mu(c)$$

This indicates that we might construct a finite Borel measure from a function F with the right properties.

Def A function $F : \mathbb{R} \to \mathbb{R}$ is called a distribution function if F is monotone non-decreasing, bounded, right-cont. (on $\mathbb{R}$) and $\lim_{x \to -\infty} F(x) = 0$

Then we have:

[MW Thm 6.3] **Thm** Suppose F is a distribution function on $\mathbb{R}$. Then there is a unique finite Borel measure μ_F having F as its distr. function (i.e. such that $F = F_{\mu_F}$)

μ_F is called the Leb.-Stieltjes measure corresponding to F

One sometimes write $\int_E f(x)\,dF(x)$ instead of $\underbrace{\int_E f\,d\mu_F}_{\text{the Leb. St. integral of } f \text{ over } E \text{ wrt. } F}$ ($E \in \mathcal{B}$) (f μ_F-int.)

Ex 4 Assume $g \in \mathcal{L}^1(\mathbb{R}, \mathcal{B}, \lambda)$, $g \geq 0$. We can then form the Borel measure μ on $(\mathbb{R}, \mathcal{B})$ given by "$d\mu = g\,d\lambda$", i.e. $\mu(E) = \int_E g\,d\lambda$ for all $E \in \mathcal{B}$

(and then we have $\int_E f\,d\mu = \int_E fg\,d\lambda$). Note that $\mu(\mathbb{R}) = \int_{\mathbb{R}} g\,d\lambda < \infty$ by ass.

We have that $F_\mu(x) = \int_{(-\infty, x]} g\,d\lambda$, $x \in \mathbb{R}$.

Hence the thm gives that ~~and~~ μ is the Leb. Stieltjes measure μ_F corresp. to $F := F_\mu$.

we have ~~and~~ $\int_E f(x)\,dF(x) = \int_E f(x)\,g(x)\,d\lambda(x)$ $\forall f$ Borel meas. non-neg.

Sketch of the proof of the Thm :

We define $\ell_F : \mathcal{J} \to [0,\infty)$ by

$$\begin{cases} \ell_F((a,b]) = F(b) - F(a) & \text{if } -\infty \leq a \leq b < \infty \\ \ell_F((c,\infty)) = F(\infty) - F(c) & \text{if } -\infty \leq c < \infty \end{cases}$$

where $F(-\infty) := 0$ and $F(\infty) := \lim_{x\to\infty} F(x) \in \mathbb{R}^+$.

Then one shows that ℓ_F is a premeasure on $\mathcal{J}$ $(*)$.

By the Carathéodory ext. thm. [HW Thm 6.1], there exists a Borel measure μ_F on $\mathcal{B} = \sigma(\mathcal{J})$ which extends ℓ_F.
Moreover, since $\mathbb{R} = (-\infty,\infty) \in \mathcal{J}$ and $\ell_F(\mathbb{R}) = F(\infty) < \infty$,
Thm. 6.2 says that μ_F is uniquely determined as an ext. of ℓ_F.

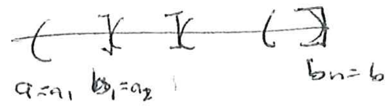
About $(*)$:

Assume $C_1, \dots, C_n \in \mathcal{J}$ (pairwise disj.) and $C := \bigcup_{k=1}^n C_k \in \mathcal{J}$.

Then $\ell_F\left(\bigcup_{k=1}^n C_k\right) = \sum_{k=1}^n \ell_F(C_k)$.

For example if $C = (a,b]$ $-\infty < a < b < \infty$, may assume $C_k = (a_k, b_k]$

where $a = a_1 < b_1 = a_2 < b_2 = a_3 < \dots < b_n = b$



So

$$\ell_F(\underbrace{(a,b]}_{\bigcup_k C_k}) = F(b) - F(a) = F(b_n) - F(a_1)$$
$$= F(b_n) - F(a_n) + F(b_{n-1}) - F(a_{n-1}) + \dots + F(b_1) - F(a_1)$$
$$= \sum_{k=1}^{\infty} \ell_F(\underbrace{(a_k, b_k]}_{C_k})$$

$$\boxed{\begin{array}{l}\text{Suppose } \{C_k\}_{k\geq 1} \subseteq \mathcal{J},\ C \in \mathcal{J} \text{ and } C \subseteq \bigcup_{k=1}^{\infty} C_k \\ \text{Then } \lambda_F(C) \leq \sum_{k=1}^{\infty} \lambda_F(C_k)\end{array}}$$

~~For example~~ Note: $C = (a,b]$, ~~$C_k = (a_k, b_k]$~~ $(a,b] = \bigcup$

[The proof is similar to the one given for the construction of Lebesgue measure on $\mathbb{R}$, but one has to exploit the right-continuity of F in a suitable way]

Read the details yourself!

Note that we can easily compute μ_F for evry intervall : For example

Let $a, b \in \mathbb{R}$, $a < b$. Then

$$\mu_F((a,b)) \underset{\substack{\uparrow \\ \delta = b-a}}{=} \mu_F\Big(\bigcup_{n\in\mathbb{N}} (a, b-\tfrac{\delta}{n}]\Big) = \lim_{n\to\infty} \mu_F\big((a, b-\tfrac{\delta}{n}]\big)$$

$$= \lim_{n\to\infty} F(b-\tfrac{\delta}{n}) - F(a) = F(b^-) - F(a)$$

where $F(b^-) := \lim_{x\to b^-} F(x)$

and

$$\mu_F(\{b\}) = \mu_F((a,b] \setminus (a,b))$$
$$= F(b) - F(a) - (F(b^-) - F(a))$$
$$= F(b) - F(b^-) \qquad \text{"the jump of } F\text{" at } b$$

In part. $\mu_F(\{b\}) = 0 \iff F(b) = F(b^-)$

$\iff F$ er venstre kont. i b

$\iff F$ er kont. i b