11th October, 2022

COURSE MAT4410

Mandatory assignment 1 of 1

Submission deadline

Thursday 27th October 2022, 14:30 in Canvas (canvas.uio.no).

Instructions

Note that you have **one attempt** to pass the assignment. This means that there are no second attempts.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) no later than the same day as the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

To pass the assignment you need to answer correctly 50 % of it.

Problem 1. (10 points) Let λ be the Lebesque measure on $\mathbb{R}$ with its σ-algebra $\mathcal{B}$ of Borel sets. Let $F : \mathbb{R} \to \mathbb{R}$ be given by

$$F(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 3x^{\frac{1}{3}}, & \text{if } 0 < x \leq 1 \\ 3, & \text{if } x > 1. \end{cases}$$

Show that F is a distribution function and find the associated Borel measure μ.

Problem 2. (10 points) (The polar decomposition of a complex measure.) Let ν be a complex measure on a measurable space $(X, \mathcal{A})$. It is known that its total variation $|\nu|$ is a finite measure on $(X, \mathcal{A})$. Prove that there exists an $\mathcal{A}$-measurable function $\phi : X \to \mathbb{C}$ such that $|\phi| = 1$ for $|\nu|$-a.e. x and

$$\nu(A) = \int_A \phi \, d|\nu|, \text{ for all } A \in \mathcal{A}.$$

Hint: use the result in Exercise 9.52 from McDonald and Weiss.

Problem 3. (a) (5 points) Let Ω be a nonempty set and $\mathcal{C}$ a countable set of pairwise disjoint subsets of Ω such that $\Omega = \bigcup_{E \in \mathcal{C}} E$. Let $\sigma(\mathcal{C})$ be the σ-algebra generated by $\mathcal{C}$ and let $f : \Omega \to \mathbb{R}$ be $\sigma(\mathcal{C})$-measurable. Prove that

$$f = \sum_{E \in \mathcal{C}} a_E \chi_E$$

for some real numbers a_E and $E \in \mathcal{C}$.

(b) (5 points) Let $(\Omega, \mathcal{A}, P)$ be a probability space and suppose that $F_1, \ldots, F_n$ are pairwise disjoint sets in $\mathcal{A}$ such that $\Omega = \bigcup_{j=1}^n F_j$ and $P(F_j) > 0$ for all $j = 1, \ldots, n$. Let $\mathcal{G} = \sigma(\{F_1, \ldots, F_n\})$ and let $f \in \mathcal{L}^1(\Omega, \mathcal{A}, P)$, with f real valued. Prove that $\mathcal{E}(f \mid \mathcal{G})$, the conditional expectation of f given $\mathcal{G}$, is of the form

$$\mathcal{E}(f \mid \mathcal{G}) = \sum_{j=1}^n a_j \chi_{F_j}$$

for suitable scalars $a_j \in \mathbb{R}$, $j = 1, \ldots, n$. Find these.

Problem 4. Let $(\Omega, \mathcal{A}, P)$ be a probability space and suppose $\mathcal{G}$ is a σ-algebra of subsets of Ω contained in $\mathcal{A}$. For a complex-valued function $f \in \mathcal{L}^1(\Omega, \mathcal{A}, P)$ we let $\mathcal{E}(f \mid \mathcal{G})$ be the conditional expectation of f given $\mathcal{G}$.

(a) (5 points) Show that the map $\mathcal{E}(\cdot \mid \mathcal{G}) : L^1(\Omega, \mathcal{A}, P) \to L^1(\Omega, \mathcal{G}, P|_{\mathcal{G}})$ sending f to $\mathcal{E}(f \mid \mathcal{G})$ is linear. Show that $\mathcal{E}(f \mid \mathcal{G}) \geq 0$ whenever $f(x) \geq 0$ for all $x \in \Omega$.

(b) (10 points) Show that $|\mathcal{E}(f \mid \mathcal{G})| \leq \mathcal{E}(|f| \mid \mathcal{G})$ P-a.e. for every $f \in L^1(\Omega, \mathcal{A}, P)$. Hint: You may assume first that f is real valued. In the case of complex valued f, consider the sign function $\rho = \text{sign}(\overline{\mathcal{E}(f \mid \mathcal{G})})$, and use the fact that ρ can be approximated by simple $\mathcal{G}$-measurable functions ρ_n with $|\rho_n| \leq 1$ for P-a.e. x.

(c) (5 points) Conclude that $\|\mathcal{E}(f \mid \mathcal{G})\|_1 \leq \|f\|_1$ for all $f \in L^1(\Omega, \mathcal{A}, P)$.

Problem 5. (10 points) Let X be a normed space. A linear operator $P : X \to X$ is a *projection* if both its kernel $P^{-1}(\{0\})$ and range $P(X)$ are closed, and $P^2 = P$ (by P^2 we mean $P \circ P$).

Show that if X is a Banach space and P is a projection on X, then P is continuous. Show moreover that

$$\|f - P(f)\| \geq \text{dist}(f, P(X))$$

for each $f \in X$, where $\text{dist}(f, P(X))$ is the distance from f to $P(X)$, computed in the metric associated to the norm.

Problem 6. (This exercise shows that there exist continuous functions whose Fourier transform $S_n(f)$ diverges at some point.)

Let X be the Banach space $C[-\pi, \pi]$ of complex valued functions with supremum norm $\|f\|_\infty = \sup\{|f(x)| \mid x \in [-\pi, \pi]\}$. For k an integer, let e_k be the function in X given by $e_k(x) = (2\pi)^{-1/2} e^{ikx}$. Define the k-th Fourier coefficient of f in X by

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx.$$

For each $n \geq 1$, let $S_n : X \to X$ be the linear operator which assigns to a function f its Fourier series

$$S_n(f) = \sum_{k=-n}^{n} \hat{f}(k) \sqrt{2\pi} e_k.$$

(a) (10 points) Show that S_n is a projection with range $\text{span}\{e_k \mid -n \leq k \leq n\}$ in the sense of Problem 5. Deduce that for $S_n(f)$ to converge uniformly to f it must hold that $f(\pi) = f(-\pi)$.

(b) (10 points) Prove that for all $n \geq 1$ and $f \in X$ we have

$$S_n(f)(x) = \frac{1}{2} \int_{-\pi}^{\pi} f(t) \frac{\sin((n + 1/2)(x - t))}{\sin((x - t)/2)} dt.$$

(c) (10 points) Let W be the closed subspace of functions in X such that $f(\pi) = f(-\pi)$. Show that

$$\sup\{|S_n(f)(x)| \mid f \in W, \|f\| \leq 1\} = \frac{1}{2}\int_{-\pi}^{\pi} |\frac{\sin((n+1/2)(x-t))}{\sin((x-t)/2)}|dt.$$

Deduce that $\sup_{n\geq 1} \|S_n\| = \infty$.

Hint: To show that the right hand side is dominated by the left hand side, use part (b) of this problem and make use of the sign of the function $D_n(x-t) = \frac{\sin((n+1/2)(x-t))}{\sin((x-t)/2)}$.

(d) (10 points) Conclude that there exists f whose Fourier series is not uniformly convergent. (In fact, it can be shown that there exists f whose Fourier series $S_n(f)$ diverges at some point, for example at $x = 0$.)