MAT4410 - Lecture 4, 31 August 2022

This note gives an example showing that Tonneli's theorem does not apply when one of the measures fails to be σ-finite.

Let $\mathcal{B}$ be the σ-algebra of Borel subsets of $[0,1]$, and let λ be Lebesgue measure and γ be the counting measure on $\mathcal{B}$, respectively. If $D = \{(x,x) \mid x \in [0,1]\}$ in $[0,1] \times [0,1]$, then Tonneli's theorem fails for $f = 1_D$ by computing, as shown in class, that

(1) $\int_{[0,1]} \int_{[0,1]} f(x,y) d\lambda(x) d\gamma(y) = 0$;
(2) $\int_{[0,1]} \int_{[0,1]} f(x,y) d\gamma(y) d\lambda(x) = 1$;

It is possible to show that $\int_{[0,1]\times[0,1]} f(x,y) d(\lambda \times \gamma)(x,y) = \infty$. (Try it.) One possibility is to argue as follows: use the definition of the product measure as an outer measure

$$(\lambda \times \gamma)(D) = \inf \big\{ \sum_n \lambda(A_n)\gamma(B_n) \mid D \subset \bigcup_n A_n \times B_n \big\};$$

for a given cover $\bigcup_n A_n \times B_n$ note that $[0,1] \subset \cup_n B_n$ then choose the finite sets among the B_n's, let F be their union, estimate $\lambda([0,1] \setminus F)$, and conclude that there must be a set $A_{n_k} \times B_{n_k}$ with B_{n_k} infinite and A_{n_k} of nonzero measure.