MAT4410, oral exam 16 December 2022

Questions at the exam will be taken from the following list. You can expect to be asked questions from all five parts. The references are to the book "A Course in Real Analysis", 2nd Edition, by John McDonald and Neil A. Weiss, hereafter referred to as [McDW], and to the book "Spaces" by Tom Lindstrøm, hereafter [L].

Product measures

(1) The semi-algebra of measurable rectangles with the premeasure constructed as product of two given measures. Examples. See [L, Thm 8.7.2] or Lecture 1.
(2) Iterated integrals for nonnegative measurable functions on a product measure space, see [L, Section 8.8] or Lectures 2 and 3. The fundamental result about monotone classes, [L, Theorem 8.8.2] without proof. Tonelli's theorem for σ-finite measure spaces, with proof in the case of a characteristic function. See Lectures 2 and 3.
(3) Iterated integrals for integrable functions on the product of two measure spaces and Fubini's theorem, without proof. See [L, Section 8.8] or Lecture 4.

Lebesgue-Stieltjes measures

(1) Finite Borel measures and distribution functions: definitions, examples. See [McDW, Section 6.2] or Lecture 5.
(2) The relation between finite Borel measures and distribution functions, [McDW, Theorem 6.3], without proof.

Signed and complex measures

(1) Signed measures, definition and examples, [McDW, Section 9.1] or Lecture 7.
(2) The Hahn-decomposition of signed measures(without proof), the Jordan decomposition of signed measures, examples. See [McDW, Section 6.1] or Lecture 8.
(3) The Radon-Nikodym theorem for σ-finite measures, with proof in the case of finite measure spaces. See [McDW, Section 9.2] or Lectures 8 and 9.
(4) The total variation of a signed measure and measurable partitions. See [McDW, Section 9.3] or Lecture 9.
(5) Complex measures: definition and their total variation. See [McDW, Section 9.3] or Lectures 9 and 10.
(6) The Radon-Nikodym theorem in the case of a complex measure and a σ-finite measure. See [McDW, Section 9.3] or Lecture 11.
(7) Probability space, random variable, the expectation of a random variable, the conditional expectation of a random variable, the probability distribution of a random variable: definitions, examples. See Lecture 11.
(8) The conditional expectation given a σ-subalgebra in a probability space. Lecture 11.

Theorems about bounded operators on Banach spaces and Hilbert spaces

(1) Banach spaces and linear operators between Banach spaces, examples. See [McDW, Section 13.1] or Lecture 13.
(2) The open mapping theorem, without proof. [McDW, Section 14.2] or Lecture 13.

(3) The principle of uniform boundedness for linear operators between Banach spaces, the inverse mapping theorem and the closed graph theorem. See [McDW, Section 14.2] or Lecture 14.
(4) Projections on Banach spaces. See exercises 14.20 and 14.21 in [McDW] or problem 5 from the assignment.
(5) The Hahn-Banach extension theorem for real vector spaces, proof of the one step extension of enlarging a subspace with a vector in its complement. See [McDW, Section 14.1] or Lecture 15.
(6) The Hahn-Banach extension theorem for complex vector spaces. See [McDW, Section 14.1] or Lecture 16.
(7) Applications of the Hahn-Banach theorem to bounded linear functionals on normed spaces. Separation of a point and a closed subspace. The distance from a point to a closed subspace of a normed space. See [McDW, Section 14.1] or Lecture 16.
(8) The dual of a normed space, reflexive normed spaces, reflexivity for Hilbert spaces (without proof). See [McDW, Section 14.1] or Lecture 16.

Riesz Representation theorems

(1) Bounded linear functionals on $L^p(X, \mathcal{A}, \mu)$ for $(X, \mathcal{A}, \mu)$ a measure space and $1 < p < \infty$. See [McDW, Section 13.4] or Lecture 17.
(2) The dual of $L^p(X, \mathcal{A}, \mu)$ for μ a finite positive measure and $1 < p < \infty$. See [McDW, Section 13.4] or Lecture 18. Reflexivity for L^p spaces, without proof, Lecture 20.
(3) The dual of $L^1(X, \mathcal{A}, \mu)$ for μ a σ-finite positive measure, without proof. See Lecture 19.
(4) Regular Borel measures on Ω for Ω a compact or locally compact metric (or topological) space, definition. The spaces $M_+(\Omega)$, $M_r(\Omega)$ and $M(\Omega)$ of all finite, signed and respectively complex regular Borel measures for Ω compact or locally compact. See [McDW, Section 13.5] or Lecture 21.
(5) The bounded linear functional on $C(\Omega) = C(\Omega, \mathbb{C})$ associated to a regular Borel (complex) measure when Ω is a compact Hausdorff space. See [McDW, Section 13.5] or Lecture 21.
(6) Positive linear functionals on $C(\Omega)$. Lecture 21.
(7) Jordan decomposition of positive linear functionals on $C(\Omega)$ that map real-valued functions to real numbers, without proof. See [McDW, Section 13.5] or Lecture 22.
(8) The Riesz-Markov theorem for positive linear functionals on $C(\Omega)$ with Ω a compact Hausdorff space, with proof of the existence of the positive regular Borel measure associated to the functional. See [McDW, Section 13.5] or Lecture 23 (of 8 November).
(9) The dual of $C(\Omega)$ for Ω compact Hausdorff, see [McDW, Section 13.6] or Lecture 24.
(10) The space $C_0(\Omega)$ for a locally compact Hausdorff space Ω. The dual of $C_0(\Omega)$, without proof. See [McDW, Section 13.6] or Lecture 25.
(11) Evaluation functionals on $C_0(\Omega)$ and the dual of $c_0(\mathbb{N})$. Lecture 25.

References

[McDW] John McDonald and Neil A. Weiss, A Course in Real Analysis, 2nd Edition, Academic Press.
[L] T. Lindstrøm, Spaces, American Math. Soc., Pure and applied undergraduate texts, vol. 29.