Exercise 6.61

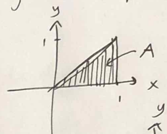
Let $X = Y = [0,1]$, $\mathcal{A} = \mathcal{B} = \mathcal{M}_{[0,1]}$, the Lebesgue σ-algebra on $[0,1]$, $\mu = \nu = \lambda_{[0,1]}$ and suppose that $f \in \mathcal{L}^1(\mu \times \nu)$

We need to prove that

$$\underbrace{\int_0^1 \left[\int_0^x f(x,y)\, dy \right] dx}_{(i)} = \underbrace{\int_0^1 \left[\int_y^1 f(x,y)\, dx \right] dy}_{(ii)}$$

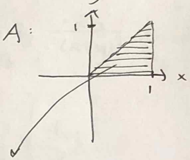
We notice first that (i) can be rewritten as

$$\int_0^1 \int_0^1 f(x,y)\, \chi_A \, dy\, dx \quad \text{with } A = \{ (x,y) \in [0,1]^2 \mid 0 \le y \le x \le 1 \}$$



and that (ii) can be rewritten as

$$\int_0^1 \int_0^1 f(x,y)\, \chi_A \, dx\, dy \quad \text{with the same } A:$$



$\{ (x,y) \mid 0 \le y \le x \le 1 \}$

By doing this we have unified the limits in (i) and (ii), and if we are able to use Fubini then we have proved the desired equality.

Check if Fubini can be used:

$f \in \mathcal{L}^1(\mu \times \nu)$ so f is $\mu \times \nu$-measurable, the same holds for χ_A, hence their product is $\mu \times \nu$-measurable and we have that

$$\iint_{[0,1]^2} |f(x,y)\, \chi_A| \, d(\mu \times \nu)(x,y) = \iint_{[0,1]^2} |f(x,y)|\, \chi_A \, d(\mu \times \nu)$$

$$\le \iint_{[0,1]^2} |f(x,y)| \, d(\mu \times \nu) < \infty .$$

Thus, condition (i) in Fubini's is satisfied and we can use Fubini's to ~~switched~~ switch the order of the ~~inset~~ integrals, which yields that

$$\int_0^1\int_0^x f(x,y)\,dy\,dx = \int_0^1\int_0^1 f(x,y)\,\chi_A\,dy\,dx$$

$$\overset{\text{Fub}}{=} \int_0^1\int_0^1 f(x,y)\,\chi_A\,dx\,dy$$

$$= \int_0^1\int_y^1 f(x,y)\,\cancel{\chi_A}\,dx\,dy.$$

Exercise 6.63

a) The Borel σ-alg. $\mathcal{B}_2$ on $\mathbb{R}^2$ is the σ-algebra generated by the open sets of $\mathbb{R}^2$ and a function $F: \mathbb{R}^2 \to \mathbb{R}$ is $\mathcal{B}_2$-measurable if $F^{-1}(O) \in \mathcal{B}_2$ for all open sets $O \in \mathbb{R}$.

Let $g: \mathbb{R}^2 \to \mathbb{R}$ be the function given by $g(x,y) = x-y$. [Then g is continuous, hence the ~~pree~~ pre-image of every open set is open, thus g is $\mathcal{B}_2$-measurable] Next, we see that $F(x,y) = f(g(x,y))$ and hence $F^{-1}(O) = g^{-1}(f^{-1}(O))$, since f is $\mathcal{B}$-measurable we have that $f^{-1}(O) \in \mathcal{B}$. It remains to show that ~~the pre-image of a Borel set B under a continuous function is a Borel set, that is~~ $g^{-1}(B) \in \mathcal{B}_2$. We do this by showing that the pre-image of a Borel set B under a continuous function is a Borel set.

Let $h: X \to Y$ be continuous and let

$$S = \{ A \subseteq Y \mid h^{-1}(A) \in \mathcal{B}_X \}$$

We show that S is a σ-algebra containing all the open sets in Y. Indeed, since ~~f~~ h is cont. $h^{-1}(O)$ is open for all open sets $O \subseteq Y$, thus $O \in S$ $\forall$ open sets $O \subseteq Y$.

Showing that S is a σ-algebra:

- non-emty when X is non-empty.
- closed under complements: Let $A \in S$, then

$$h^{-1}(A^c) = h^{-1}(Y \setminus A) = \{x \in X \mid h(x) \notin A\}$$
$$= h^{-1}(A)^c \in \mathcal{B}_X \text{, since } \mathcal{B}_X$$

is a σ-alg.

- Closed under countable union:
  Let $A_1, \dots, A_n$ be a countable collection of sets in S, then
  $h^{-1}(\bigcup_n A_n) = \bigcup_n h^{-1}(A_n) \in \mathcal{B}_X$, hence $\bigcup_n A_n \in \mathcal{B}_X$

Hence, S is a σ-alg. and since $\mathcal{B}_Y$ is the minimal σ-alg. containing all the open sets of Y, we must have that $\mathcal{B}_Y \subseteq S$. Thus, we can conclude that $h^{-1}(B) \in \mathcal{B}_X$ for all borel-sets $B \subseteq Y$.
In particular, $g^{-1}(B) \in \mathcal{B}_2$, thus
$F^{-1}(O) = g^{-1}(f^{-1}(O)) \in \mathcal{B}_2$ for all open sets $O \in \mathcal{B}$.
Hence, F is $\mathcal{B}_2$-measurable.

b) $\xi: \mathbb{R} \to \mathbb{R}$ $\mathcal{B}$-measurable and $\varphi: \mathbb{R}^2 \to \mathbb{R}$ is given by $\varphi(x,y) = \xi(y)$. Let $O \subseteq \mathbb{R}$ be open, then
$\varphi^{-1}(O) = \{(x,y) \mid \varphi(x,y) \in O\} = \{(x,y) \mid \xi(y) \in O\}$, and
$\xi^{-1}(O) = \{y \mid \xi(y) \in O\}$. Hence $\varphi^{-1}(O) = \mathbb{R} \times \xi^{-1}(O)$.
Now, $\xi^{-1}(O) \in \mathcal{B}$, since ξ is $\mathcal{B}$-measurable.
It remains to show that $\mathbb{R} \times \xi^{-1}(O) \in \mathcal{B}_2$. We do this by showing that
$S_* = \{F \subseteq \mathbb{R} \mid \mathbb{R} \times F \in \mathcal{B}_2\}$ is a σ-algebra containing all the open sets. S_* contains all the open sets since the cartesian product of $\mathbb{R} \times O$ is open in $\mathbb{R}^2$ for all open $O \subseteq \mathbb{R}$. S_* is clearly not empty and S_* is closed under complements since $(\mathbb{R} \times F^c) = (\mathbb{R} \times F)^c \in \mathcal{B}_2$ at last it is closed under countable unions, since $\mathbb{R} \times \bigcup_n A_n = \bigcup_n \mathbb{R} \times A_n \in \mathcal{B}_2$

~~$(\mathbb{R}\times E)^c = \mathbb{R}\times E^c$~~

Hence S_x is a σ-algebra containing all open sets in $\mathbb{R}$, thus by the minimality of the Borel σ-alg. $\mathcal{B}$ we must have that $\mathcal{B} \subseteq S_x$. Hence $\mathbb{R}\times B \in \mathcal{B}_2$ for all $B\in\mathcal{B}$, and in particular $\mathbb{R}\times \xi^{-1}(O)\in\mathcal{B}_2$, thus φ is $\mathcal{B}_2$-measurable.

~~c) lack of paper~~

d) Next page ~~Let $f, g \in \mathcal{L}^1(\mathbb{R},\mathcal{B},\lambda)$.~~ We bootstrap over g.

Suppose first that $g = \chi_A$, $A\in\mathcal{B}$, then

$$(f*g)(x) = \int_{\mathbb{R}} f(x-y)\, g(y)\, d\lambda(y)$$

$$= \int_{\mathbb{R}} f(x-y)\,\chi_A\, d\lambda(y)$$

$$\le \int_{\mathbb{R}} |f(x-y)|\,\chi_A\, d\lambda(y)$$

$$\le \int_{\mathbb{R}} |f(x-y)|\, d\lambda(y) < \infty, \text{ since } f\in\mathcal{L}^1(\mathbb{R}, \mathcal{B},\lambda).$$

Next, let g be a simple function. Then $g = \sum_{i=1}^n a_i \chi_{A_i}$ and we get that

$$(f*g)(x) = \int_{\mathbb{R}} f(x-y)\, g(y)\, d\lambda(y)$$

$$= \sum_i a_i \int_{\mathbb{R}} f(x-y)\,\chi_{A_i}\, d\lambda \le \sum_{i=1}^n a_i \int_{\mathbb{R}} |f(x-y)|\, d\lambda < \infty$$

At last, let g be the pointwise limit of

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c) $h : \mathbb{R} \to [0,\infty]$ is $\mathcal{B}$-measurable.

Assume first that h is a ~~simple~~ indicator function, that is $h = \chi_A$ for some $\mathcal{B}$-measurable set A.

Then, for an arbitrary ~~y~~ fixed $y \in \mathbb{R}$,

$$\int_{\mathbb{R}} h(x-y)\, d\lambda(x) = \int_{\mathbb{R}} \chi_A(x-y)\, d\lambda(x)$$

$$= \mu(\{x+y \mid x \in A\})$$

$$= \mu(A) = \int_{\mathbb{R}} \chi_A(x)\, dx$$

Assume now that h is a simple function, that is $h = \sum_{i=1}^{n} a_i \chi_{A_i}$ for $a_i \in \mathbb{R}$ and A_i $\mathcal{B}$-measurable. Then

$$\int_{\mathbb{R}} h(x-y)\, d\lambda(x) = \int_{\mathbb{R}} \sum_{i=1}^{n} a_i \chi_{A_i}(x-y)\, d\lambda(x)$$

$$= \sum_{i=1}^{n} a_i \int_{\mathbb{R}} \chi_{A_i}(x-y)\, d\lambda(x)$$

$$= \sum_{i=1}^{n} a_i \int_{\mathbb{R}} \chi_{A_i}(x)\, d\lambda(x)$$

$$= \int_{\mathbb{R}} h(x)\, d\lambda(x)$$

At last, since $h : \mathbb{R} \to [0,\infty]$ is non-negative it is ~~the~~ the pointwise limit of a monotonic increasing sequence of non-negative simple functions. Thus the final result follows by applying the monotone convergene theorem.

~~d) Let $F(x,y) = f(x-y)$ and $\varphi(x,y) = g(y)$, then~~

~~$(f * g)(x) = \int_{\mathbb{R}} [F \cdot \varphi]_{[x]}(y) \, d\lambda(y)$.~~

d) We want to use Fubinis theorem to prove this part. Indeed let $h(x,y) = F(x,y) \cdot \varphi(x,y)$ where $F(x,y) = f(x-y)$ and $\varphi(x,y) = g(y)$. Then $F(x,y)$ and $\varphi(x,y)$ are $\mathcal{B}_2$-measurable by parts a) and b), thus their product h is also $\mathcal{B}_2$-measurable. Moreover, we have that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |h(x,y)| \, d\lambda(x) \, d\lambda(y)$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x-y)| \, |g(y)| \, d\lambda(x) \, d\lambda(y)$$

$$= \int_{\mathbb{R}} |g(y)| \left[\int_{\mathbb{R}} |f(x-y)| \, d\lambda(x) \right] d\lambda(y)$$

$$= \int_{\mathbb{R}} |g(y)| \left[\int_{\mathbb{R}} |f(x)| \, d\lambda(x) \right] d\lambda(y)$$

$$= \int_{\mathbb{R}} |f(x)| \, d\lambda(x) \cdot \int_{\mathbb{R}} |g(y)| \, d\lambda(y) < \infty$$

since $f, g \in \mathcal{L}^1(\mathbb{R}, \mathcal{B}, \lambda)$. Hence condition (ii) is satisfied, ~~ca Fubini~~ and we can use Fubini. By part c) (or d)) we have that

$g(x) = \int_{\mathbb{R}} \cancel{f} \, h(x,y) \, d\lambda(y)$ is defined μ.a.e and is in $\mathcal{L}^1(\mathbb{R}, \mathcal{B}, \lambda)$

$= \int_{\mathbb{R}} f(x-y) g(y) \, d\lambda(y) = f * g(x)$

Exercise 8.8.8

$f : [0,1] \times [0,1] \to \mathbb{R}$, $f(x,y) = \dfrac{x-y}{(x+y)^3}$ for $(x,y) \neq (0,0)$

and ~~ftxp~~ $f(0,0) = c$ for some $c \in \mathbb{R}$.

This function is Rieman integrable on $[0,1] \times [1/n, 1]$ and $[1/n, 1] \times [0,1]$ for ~~all~~ each $n \in \mathbb{N}$ (it is bounded and continuous) and we see that

$$\int_{1/n}^{1}\int_{0}^{1} f(x,y)\,dx\,dy = \int_{1/n}^{1}\int_{0}^{1} \frac{x-y}{(x+y)^3}\,dx\,dy$$

$$= \int_{1/n}^{1}\int_{0}^{1} \frac{x+y-2y}{(x+y)^3}\,dx\,dy$$

$$= \int_{1/n}^{1}\left[\int_{0}^{1} \frac{x+y}{(x+y)^3}\,dx + \int_{0}^{1} \frac{-2y}{(x+y)^3}\,dx\right]dy$$

$$= \int_{1/n}^{1}\left[\left[\frac{-1}{(x+y)}\right]_0^1 + \left[\frac{y}{(x+y)^2}\right]_0^1\right]dy$$

$$= \int_{1/n}^{1} \frac{-1}{1+y} + \frac{1}{y} + \frac{y}{(1+y)^2} - \frac{1}{y}\,dy$$

$$= \int_{1/n}^{1} \frac{-1}{(1+y)^2}\,dy = \left[\frac{1}{y+1}\right]_{1/n}^{1} = \frac{1}{2} - \frac{n}{1+n}$$

Similarly, we get that

$$\int_{1/n}^{1}\int_{0}^{1} f(x,y)\,dy\,dx = \int_{1/n}^{1}\int_{0}^{1} \frac{x-y}{(x+y)^3}\,dy\,dx = -\int_{1/n}^{1}\int_{0}^{1} \frac{(y-x)}{(x+y)^3}\,dy\,dx$$

$$= -\int_{1/n}^{1} \frac{-1}{(1+x)^2}\,dx = \left[\frac{-1}{x+1}\right]_{1/n}^{1} = \frac{-1}{2} + \frac{n}{1+n}$$

~~Now let $f_n^1(x,y) = f(x,y)\cdot \chi_{[0,1]\times[1/n,1]}$ and let $f_n^2(x,y) = f(x,y)\chi_{[1/n,1]\times[0,1]}$. Then both $\{f_n^1\}_{n\in\mathbb{N}}$ and $\{f_n^2\}_{n\in\mathbb{N}}$ are~~

Now let $f_n(x) = \frac{1}{(x+1)^2}\chi_{[1/n,1]}$ for each $n\in\mathbb{N}$. Then $\{f_n\}_{n\in\mathbb{N}}$ is a monotone non-decreasing sequence of lebesgue measurable functions on $[0,1]$ that converges pointwise to $\frac{1}{(x+1)^2}$ on $[0,1]$.

~~Let now $f_n(x) = g(x)\chi_{[1/n,1]}$ for each $n\in\mathbb{N}$, where~~

$$\cancel{g(x) = \begin{cases} \frac{1}{1+x} & \text{if } x\in(0,1] \\ c & \text{if } x=0 \end{cases}}$$

Hence, by the monotone convergence theorem

$$\cancel{\lim_{n\to\infty}\int_{1/n}^{1}}$$

$$\lim_{n\to\infty}\int_{1/n}^{1}\frac{1}{(1+x)^2}\,dx = \lim_{n\to\infty}\int_0^1 \frac{1}{(1+x)^2}\chi_{[1/n,1]}\,dx$$

$$= \int_0^1 \lim_{n\to\infty}\frac{1}{(1+x)^2}\chi_{[1/n,1]}\,dx$$

$$= \int_0^1 \frac{1}{(x+1)^2}\,dx = \frac{1}{2}$$

By a similar argument we get that

$$\lim_{n\to\infty} \int_{1/n}^{1} \frac{-1}{(1+y)^2}\, dy = -\lim_{n\to\infty} \int_{1/n}^{1} \frac{1}{(1+y)^2}\, dy = -1/2.$$

This gives us that

$$\int_0^1\int_0^1 f(x,y)\, dx\, dy = \lim_{n\to\infty} \int_{1/n}^{1} \frac{-1}{(1+y)^2}\, dy = -\frac{1}{2}$$

and that

$$\int_0^1\int_0^1 f(x,y)\, dy\, dx = \lim_{n\to\infty} \int_{1/n}^{1} \frac{1}{(1+x)}\, dx = 1/2$$

Does it hold that $f \in \mathcal{L}^1(\lambda_{[0,1]} \times \lambda_{[0,1]})$?

No, because that would contradict Fubini.

Exercise 8.7.4

Let A be an open set in $\mathbb{R}^n$, and ~~a~~ let $x \in A$. Since A is open there exists an ε-ball $B_{\varepsilon_x}(x)$ centered at x s.t. $B_{\varepsilon_x}(x) \subseteq A$.

Next, we can find an n-dim rectangle $(a_1, b_1) \times \dots \times (a_n, b_n)$ with $a_i, b_i \in \mathbb{Q}$ such that $x \in \underbrace{(a_1, b_1) \times \dots \times (a_n, b_n)}_{R_x} \subseteq B_{\varepsilon_x}(x)$, and we have that

$$A = \bigcup_{x \in A} x \subseteq \bigcup_{x \in A} R_x \quad \text{and} \quad \bigcup_{x \in A} R_x \subseteq \bigcup_{x \in A} B_{\varepsilon_x}(x) \subseteq A$$

Thus $A = \bigcup_{x \in A} R_x$. At last, this union ~~is~~ can allways be written as a countable union, since $\{(a_1, b_1) \times \dots \times (a_n, b_n) \mid a_i, b_i \in \mathbb{Q}\}$ is countable.

Exercise 8.8.6

$(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{M}, \lambda)$ and $(Y, \mathcal{B}, \nu) = (\mathbb{N}, \mathcal{P}(\mathbb{N}), \nu)$ ← counting

We wish to find $\int \frac{1}{1+(2^n x)^2} \, d(\mu \times \nu)$.

We proceed by calculating the following integral

$$\int_{\mathbb{N}} \int_{-k}^{k} \frac{1}{1+(2^n x)^2} \, d\mu \, d\nu = \int_{\mathbb{N}} \left[\frac{1}{2^n} \arctan(2^n x) \right]_{-k}^{k} d\nu$$

$$= \int_{\mathbb{N}} \frac{1}{2^n} \cdot 2 \cdot \arctan(2^n k) \, d\nu$$

We are interested in the value of this integral as $k \to \infty$. Define $\{f_k\}_{k \in \mathbb{N}}$ to be the sequence

$f_k = \frac{1}{1+(2^n x)^2} \chi_{[-k,k]}$, then $\{f_k\}_{k \in \mathbb{N}}$ is a non-decreasing sequence of non-neg. functions ~~then $\{f_k\}$ is bounded from above by the Lebesgue integrable non negative function~~ ~~$g = \frac{1}{1+(2^n x)^2}$~~. So by using the ~~dominated~~ monotone convergence theorem we get that

$$\int_{\mathbb{N}} \int_{\mathbb{R}} \frac{1}{1+(2^n x)^2} \, d\mu \, d\nu = \int_{\mathbb{N}} \int_{\mathbb{R}} \lim_{k \to \infty} \frac{1}{1+(2^n x)^2} \chi_{[-k,k]} \, d\mu \, d\nu$$

$$= \int_{\mathbb{N}} \lim_{k \to \infty} \int_{-k}^{k} \frac{1}{1+(2^n x)^2} \, d\mu \, d\nu$$

$$= \int_{\mathbb{N}} \lim_{k \to \infty} \frac{1}{2^n} 2 \arctan(2^n k) \, d\nu$$

$$= \int_{\mathbb{N}} \frac{1}{2^n} \pi \, d\nu = \sum_{n=1}^{\infty} \frac{1}{2^n} \pi = \underline{\underline{\pi}}$$

Exercise 8.8.12

$(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are complete σ-finite measure spaces.

def. complete: $\forall N \subseteq X$ s.t. $N \subseteq A \in \mathcal{A}$ with $\mu(A) = 0$ we have that $N \in \mathcal{A}$ (and as a consequence $\mu(N) = 0$).

def. σ-finite: $\exists$ a sequence $\{A_n\}_{n \in \mathbb{N}}$ s.t. $\bigcup_{n \in \mathbb{N}} A_n = X$ and $\mu(A_n) < \infty$ $\forall$ n.

a) $E \in \mathcal{A} \times \mathcal{B}$ and $\mu \times \nu(E) = 0$. Then by lemma 8.8.3 ~~$\mu(E^y)$~~ We have that $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$ are measurable and that

$$0 = \mu \times \nu(E) = \int \nu(E_x)\, d\mu = \int \mu(E^y)\, d\nu$$

~~$\nu(E_x)$ and by lemma 8.8.1 we have that E_x is ν-measurable and E^y is μ-measurable.~~

Thus the function $x \mapsto \nu(E_x)$ is 0 μ-a.e and $y \mapsto \mu(E^y)$ is 0 ν-a.e. This proves the claim.

b) ~~In order to prove this claim it suffices to show that $N^y \subseteq E^y$ (Then we can use part (a) to conclude)~~ We have that

$N^y = \{x \in X \mid (x,y) \in N\} \subseteq \{x \in X \mid (x,y) \in E\} = E^y$

because $N \subseteq E$, and by lemma 8.8.1 (i) $E^y \in \mathcal{A}$ and by part a) $\mu(E^y) = 0$ for ν.a.e. y

Hence, since $(X, \mathcal{A}, \mu)$ is complete we must have that N^y is μ-measurable for ν-a.e. y and as a consequence $\mu(N^y) = 0$ for ν a.e. y. The argument is analogous for N_x.