28$^{\text{th}}$ September, 2023

COURSE MAT4410

Mandatory assignment 1 of 1

Submission deadline

Thursday 19$^{\text{th}}$ October 2023, 14:30 in Canvas (canvas.uio.no).

Instructions

Note that you have **one attempt** to pass the assignment. This means that there are no second attempts.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) no later than the same day as the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

To pass the assignment you need to answer correctly 60 % of it. A full score for a problem requires a correct and complete argumentation. The individual questions have different weights, as indicated. You may choose freely what problems from the set to solve.

Problem 1. Let λ be the Lebesque measure on $\mathbb{R}$ with its σ-algebra $\mathcal{B}$ of Borel sets. Let $F : \mathbb{R} \to \mathbb{R}$ be given by

$$F(x) = \begin{cases} 0, & \text{if } x < \frac{1}{2} \\ 1 - e^{-2x}, & \text{if } \frac{1}{2} \leq x < 1 \\ 2 - e^{-2x}, & \text{if } x \geq 1. \end{cases}$$

(a) (10 points) Show that F is nondecreasing and right continuous, so is a distribution function. Let μ be the Borel measure associated to F, and describe $\mu(A)$ on sets A from the semi-algebra consisting of intervals $(a, b]$, (c, ∞) that generates $\mathcal{B}$, where $-\infty \leq a \leq b < \infty$, $-\infty \leq c < \infty$. Compute in particular the values $\mu(\{a\})$ at jumps of F.

(b) (10 points) Define a measure on $\mathcal{B}$ by

$$\mu_1(B) = \int_B F'(t)\, d\lambda(t), B \in \mathcal{B},$$

and set $\mu_2 = (1 - e^{-1})\delta_{1/2} + \delta_1$, where δ_a is the Dirac measure at $a \in \mathbb{R}$. Explain why $\mu = \mu_1 + \mu_2$ is the Lebesgue decomposition of μ with respect to λ.

Problem 2. Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}$ and let $\mathcal{B}^2$ be the Borel σ-algebra on $\mathbb{R}^2$. Suppose that μ and ν are σ-finite Borel measures on $\mathcal{B}$.

(a) (10 points) For $E \in \mathcal{B}$, let $S = \{(x, y) \mid x + y \in E\} \subset \mathbb{R}^2$ and define functions on $\mathbb{R}$ by $g(x) = \nu(E - x)$ and $h(y) = \mu(E - y)$. Prove that $S \in \mathcal{B}^2$ and that h and g are Borel measurable. Prove moreover that

$$(\mu \times \nu)(S) = \int_{\mathbb{R}} g(x) d\mu(x) = \int_{\mathbb{R}} h(y) d\nu(y).$$

(b) (10 points) For $E \in \mathcal{B}$ we define $(\mu * \nu)(E) = \int_{\mathbb{R}} \mu(E - y) d\nu(y)$. Prove that $\mu * \nu$ is a Borel measure on $\mathcal{B}$. (It is called the convolution of μ and ν.) Show that for each complex-valued function f in $\mathcal{L}^1(\mathbb{R}, \mathcal{B}, \mu * \nu)$, we have

$$\int_{\mathbb{R}^2} f(x + y) d(\mu \times \nu)(x, y) = \int_{\mathbb{R}} f(t) d(\mu * \nu)(t).$$

Hint: Use (a) and "bootstrap".

(c) (10 points) Given a finite Borel measure μ, define the Fourier-Stieltjes transform of μ to be the function $\widehat{\mu} : \mathbb{R} \to \mathbb{C}$ given by

$$\widehat{\mu}(s) = \int_{\mathbb{R}} e^{its} d\mu(t),$$

for $s \in \mathbb{R}$. Explain why this integral is finite and prove that if ν is another finite Borel measure then $\widehat{\mu * \nu}(s) = \widehat{\mu}(s)\widehat{\nu}(s)$ for all $s \in \mathbb{R}$. (Thus the Fourier-Stieltjes transform takes convolution to pointwise product.) Hint: Fubini's theorem.

Problem 3. (Existence, linearity and boundedness of conditional expectation given a σ-subalgebra.) Let $(X, \mathcal{A}, \eta)$ be a measure space so that $\eta(X) = 1$. (Such a space is called a probability space.) Let $L^1(X, \mathcal{A}, \eta)$ be the Banach space of complex valued integrable functions on X. Suppose that $\mathcal{G}$ is a σ-algebra of subsets of X contained in $\mathcal{A}$ and let μ be the restriction of η on $\mathcal{G}$.

(a) (10 points) Show that for every $f \in L^1(X, \mathcal{A}, \eta)$ there exists a function $E_{\mathcal{G}}(f)$ in $L^1(X, \mathcal{G}, \mu)$ unique μ-a.e., such that

$$\int_G f d\eta = \int_G E_{\mathcal{G}}(f) d\mu \text{ for all } G \in \mathcal{G}.$$

Prove that the map $\Phi : L^1(X, \mathcal{A}, \eta) \to L^1(X, \mathcal{G}, \mu)$, $\Phi(f) = E_{\mathcal{G}}(f)$ is linear as a map between complex vector spaces. Prove moreover that $\Phi(f) \geq 0$ whenever $f(x) \geq 0$ for all $x \in X$. Hint: The Radon-Nikodym theorem.

(b) (20 points) Show that for each $f \in L^1(X, \mathcal{A}, \eta)$ we have

$$|\Phi(f)| \leq \Phi(|f|),\ \eta - a.e. \tag{1}$$

Conclude that

$$\|\Phi(f)\|_1 \leq \|f\|_1$$

for all $f \in L^1(X, \mathcal{A}, \eta)$.

Hint for proving the inequality in (1): Assume first that f is real valued. In the case of complex valued f, consider the sign function $\rho = \mathrm{sign}(\overline{\Phi(f)})$, and use the fact that ρ can be approximated by simple $\mathcal{G}$-measurable functions ρ_n with $|\rho_n| \leq 1$ for μ-a.e. x in X.

Problem 4. (The Cantor set and the Cantor measure, see Example 1.28 and Example 4.7 in G. Teschl's book *Topics in real analysis*.)

The *Cantor set* C is constructed as follows: let $C_0 := [0, 1]$. Then we remove the middle third of this interval, namely the open interval $(\frac{1}{3}, \frac{2}{3})$, to obtain $C_1 := [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Next, we remove the middle third from each of the two intervals in C_1 to obtain

$$C_2 := \Big[0, \frac{1}{9}\Big] \cup \Big[\frac{2}{9}, \frac{1}{3}\Big] \cup \Big[\frac{2}{3}, \frac{7}{9}\Big] \cup \Big[\frac{8}{9}, 1\Big].$$

Continuing like this to remove the middle third of each subinterval of C_n we obtain C_{n+1}, a union of closed intervals, for $n \geq 2$. The *Cantor set* is defined to be $C := \bigcap_{n \geq 0} C_n$. It can be shown that C is compact and that it does not contain any intervals.

(a) (10 points) Explain why each $x \in C$ has a unique expansion $\sum_{k=1}^{\infty} a_k 3^{-k}$ with $a_k \in \{0, 2\}$. This in particular implies that C is uncountable. Argue that C has Lebesgue measure 0.

(b) (10 points) Define a function $\varphi : C \to [0, 1]$ by

$$\varphi(x) = \sum_{k=1}^{\infty} \frac{a_k}{2} 2^{-k}.$$

The *Cantor function* $F : [0, 1] \to [0, 1]$ is defined by

$$F(x) = \sup\{\varphi(y) \mid y \in C, y \leq x\} \text{ for } x \in [0, 1].$$

Show that φ is nondecreasing and surjective. Show that F is nondecreasing and continuous on $[0, 1]$. (Hint: Look first at F on open intervals that have been removed from $[0, 1]$ in constructing C, then consider the possible behaviour at the points in C). Let μ_F be the associated Lebesgue-Stieltjes measure of F. Conclude that μ_F is singular with respect to Lebesgue measure.