Suggested solution to the exam in MAT4410, December 5, 2023.

Problem 1. Part 1a: it is routine to verify that ν is σ-additive and that $\nu(A) =$ whenever $\mu(A) = 0$. Recall that we defined the Jordan decomposition of a signed measure ν as

$$\nu_+(A) = \frac{|\nu|(A) + \nu(A)}{2} \qquad \text{and} \qquad \nu_-(A) = \frac{|\nu|(A) - \nu(A)}{2}.$$

Moreover, we have $\nu = \nu_+ - \nu_-$ and $\nu_+ \perp \nu_-$, and ν_+, ν_- form the smallest possible decomposition of ν this form. Now, given ν as in the problem, we have that (using a result proved even in the case of complex measures),

$$|\nu|(A) = \int_A |f| d\mu = \int_A f^+ d\mu + \int_A f^- d\mu = \nu_1(A) + \nu_2(A).$$

Since $\nu = \nu_1 - \nu_2$ with ν_+, ν_- mutually singular measures, using for example $X_+ = \{x \in X \mid f(x) \geq 0\}$, $\nu_-(X_+) = 0 = \nu_+(X \setminus X_+)$, and since $\nu_1 \geq \nu_+$, $\nu_2 \geq \nu_-$, we see that these inequalities must be equalities.

Part 1b: A Hahn-decomposition is $X_+ = (0, 2]$, $X_- = [2, 4]$. The Jordan decomposition is determined by

$$\nu_+(A) = \int_A 1_{(0,2]} x \cos(\frac{\pi x}{4}) dx, \nu_-(A) = \int_A 1_{[2,4]} x \cos(\frac{\pi x}{4}) dx.$$

Problem 2. For 2a, to show that $A = \{(x, r) \in X \times [0, \infty) \mid |f(x)| > r\}$ is measurable with respect to the product σ-algebra, note for example that

$$A = \bigcup_{a \in \mathbb{Q}} |f|^{-1}((a, \infty)) \times [0, a),$$

which is a countable union of measurable rectangles and therefore belongs to $\Sigma \otimes \mathcal{B}$. Alternatively, use the bootstrap technique: suppose that $g = 1_E$ for $E \in \Sigma$, then the corresponding set $A(1_E)$ is $E \times [0, 1)$, which is in the product σ-algebra. Extend to $g = \sum_j \alpha_j 1_{E_j}$, a simple function in standard form, then $A(f)$ is the union of $E_i \times [0, \alpha_i)$. Finally, for g nonnegative, approximate with a nondecreasing sequence s_n of simple functions that converges pointwise to g, and write $A(g)$ again as a countable union of measurable rectangles.

For 2b, define $h : X \times [0, \infty) \to [0, \infty)$ by $h(x, r) = 1_A(x, r)\phi(r)$. Since the measures are σ-finite, Tonelli's theorem applies and gives that

$$\begin{aligned}\int_{X\times[0,\infty)} h(x,r)\,d(\mu\times\lambda)(x,r) &= \int_{[0,\infty)} \phi(r)\big(\int_X 1_{A(r)}(x,r)\,d\mu(x)\big)d\lambda(r)\\ &= \int_{[0,\infty)} \phi(r)\mu(\{x\in X \mid |f(x)|>r\})d\lambda(r)\\ &= \int_{[0,\infty)} \phi(r)E_f(r)d\lambda(r)\\ &= \int_X\big(\int_{[0,\infty)} h(x,r)d\lambda(r)\big)d\mu(x)\\ &= \int_X\big(\int_{[0,\infty)} \phi(r)1_{\{r\mid 0\le r<|f(x)|)\}}d\lambda(r)\big)d\mu(x)\\ &= \int_X\big(\int_{[0,|f(x)|]} \phi(r)d\lambda(r)\big)d\mu(x)\\ &= \int_X (F\circ|f|)d\mu.\end{aligned}$$

Problem 3. This is an application of the closed graph theorem: if $x_n \to x$ and $P(x_n)\to y$, then $y \in P(X)$ since the range of P is closed, as a subspace of Y, thus $y = P(z)$ for some $z \in X$. Moreover, $x_n - P(x_n) \in \ker(P)$ for all $n \geq 1$, so its limit $x - y$ is in $\ker(P)$. Then $P(x) = P(y) = P(P(z)) = P(z) = y$, as wanted.

Problem 4. Part 4a. Let $x \in X$ and δ_x be Dirac measure. Given $A \subset X$ and U open such that $A \subset U$ we have $\inf \delta_x(U) = \delta_x(A)$, with the value being 0 or 1 depending on whether or not $x \in A$. Inner regularity is routine, as well. To show that $\operatorname{supp}(\delta_x) = \{x\}$, clearly $\delta_x(U) > 0$ for all open U containing x, so x is in the support of μ. Assume there is $y \neq x$ with $y \in \operatorname{supp}(\delta_x)$, then there are disjoint open sets U, V with $y \in V, x \in U$. Then $\mu(V) > 0$, however this is a contradiction as $x \notin V$. So the support is exactly $\{x\}$. The functional on $C_0(X)$ associated with δ_x is the evaluation $\ell_x(f) = f(x)$, for $f \in C_0(X)$: that $f(x) = \int_X f(t)d\delta_x(t)$ is true for simple functions, then for arbitrary f we can find a sequence $\{s_n\}_n$ of simple functions such that $f(t) = \lim_n s_n(t)$ uniformly on the support of f, so also $f(x) = \int_X f(t)d\delta_x(t)$, and by the uniqueness claim in the Riesz-Markov theorem we obtain that the evaluation functional ℓ_x corresponds to the measure δ_x.

For 4a, assume that μ is a finite regular measure with support the single point $\{x\}$. From 4a we know that the measure $C\delta_x$ has support $\{x\}$ for every constant $C > 0$. Since X is locally compact, there is K compact with $x \notin K$.

Given $y \in K$ we have $y \notin \text{supp}(\mu)$, so there is an open set V_x with $x \in V_x$ and $\mu(V_x) = 0$. Cover K with $\cup_{x \in K} V_x$ and extract a finite cover, then conclude that $\mu(K) = 0$. By inner regularity, we get $\mu(A) = 0$ for all A Borel set with the property that $x \notin A$. If now A is a Borel set so that $x \in A$, we get by the previous that $\mu(A \setminus \{x\}) = 0$. Since the measure is finite, $\mu(A) = \mu(\{x\})$. Let $C = \mu(\{x\})$. We have that $\mu(A) = C\delta_x(A)$ for all Borel sets A. Since also $C = \mu(X)$, we have $C > 0$.

Problem 5. Part 5a: see lectures.

Let λ be the Lebesgue measure on $[0, 1]$ and let $L^1([0, 1], \lambda)$ be the Banach space of complex-valued integrable functions on $[0, 1]$. Suppose that $\{f_k\}_{k \geq 1}$ is a sequence of elements in $L^1([0, 1], \lambda)$ and $\{a_k\}_{k \geq 1}$ is a sequence of complex numbers sequence of complex numbers such that there exists $M > 0$ with the property that

$$|\sum_{k=1}^{m} \alpha_k a_k| \leq M\big(\int_{[0,1]} |\sum_{k=1}^{m} \alpha_k f_k(x)| d\lambda(x)\big)$$

for any finite sequence of complex numbers $\alpha_1, \ldots, \alpha_m$ and any $m \geq 1$. Show that there is $g \in L^\infty([0, 1], \lambda)$ such that

$$\int_{[0,1]} f_k(x) g(x) d\lambda(x) = a_k, \ \forall k \geq 1.$$

Solution: Let $Y = \text{span}\{f_k \mid k \geq 1\}$, a linear subspace of $L^1([0, 1])$. Define $\varphi_0 : Y \to \mathbb{C}$ by

$$\varphi_0(\sum_{k=1}^{m} \alpha_k f_k) = \sum_{k=1}^{m} \alpha_k a_k$$

for arbitrary complex numbers $\alpha_1, \ldots, \alpha_m$ and $m \geq 1$. This is a linear map, as may be verified routinely (you need to fill in details), and the assumption gives that φ_0 is bounded with $\|\varphi_0\| \leq M$. By the Hahn-Banach theorem there is an extension $\varphi \in L^1([0, 1])^*$. Since $L^1([0, 1])^* \cong L^\infty([0, 1])$ by the Riesz representation theorem, there is a function $g \in L^\infty([0, 1])$ such that

$$\varphi(f) = \int_{[0,1]} f g d\lambda$$

for every $f \in L^1([0, 1])$. Hence $\int_{[0,1]} f_k(x) g(x) d\lambda(x) = \varphi_0(f_k) = a_k$ for all $k \geq 1$ because φ extends φ_0.