UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in MAT4410 — Advanced linear analysis.

Day of examination: Tuesday, December 5, 2023.

Examination hours: 15:00 – 19:00.

This problem set consists of 3 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note: You must justify all your answers.

Problem 1

1a

(10 points.) Let $(X, \mathcal{A}, \mu)$ be a measure space with μ a positive σ-finite measure and let $f : X \to \mathbb{R}$ be an integrable function. Explain why $\nu(A) := \int_A f d\mu$ for $A \in \mathcal{A}$ defines a signed measure satisfying $\nu \ll \mu$. Define positive measures by

$$\nu_1(A) = \int_A f^+ d\mu \qquad \text{and} \qquad \nu_2(A) = \int_A f^- d\mu, \text{ for } A \in \mathcal{A}.$$

Prove that $\nu = \nu_1 - \nu_2$ is the Jordan decomposition of the measure ν.

1b

(10 points.) Let μ be Lebesgue measure on the σ-algebra $\mathcal{A}$ of Borel subsets of $X = (0, 4]$ and let $f(x) = x \cos(\frac{\pi x}{4})$. Determine a Hahn decomposition $X = X_+ \sqcup X_-$ and the Jordan decomposition for the measure ν given by $\nu(A) = \int_A f(x) d\mu(x)$ for $A \in \mathcal{A}$.

Problem 2

Let (X, Σ, μ) be a σ-finite measure space and let λ be Lebesgue measure on the Borel σ-algebra $\mathcal{B}$ of subsets of $[0, \infty)$. Suppose that $\phi : [0, \infty) \to [0, \infty)$ is a function that is integrable over all compact intervals and $f : X \to \mathbb{C}$ is a Σ-measurable function.

(Continued on page 2.)

2a

(10 points.) Show that the set $S = \{(x, r) \in X \times [0, \infty) \mid |f(x)| > r\}$ is measurable with respect to the product σ-algebra $\Sigma \otimes \mathcal{B}$.

2b

(10 points.) Define $F(r) := \int_{[0,r]} \phi(s) d\lambda(s)$ for $r \in [0, \infty)$ and set $E_f(r) := \mu(\{x \in X \mid |f(x)| > r\})$. Prove that

$$\int_X (F \circ |f|) d\mu = \int_{[0,\infty)} \phi(r) E_f(r) d\lambda(r).$$

Problem 3

(20 points) Let X be a Banach space, let $f_1, f_2, \ldots, f_m \in X$ and let Y be the subspace of X spanned by $f_1, \ldots, f_m$. Suppose that $P : X \to Y$ is a linear map so that $x - P(x) \in \ker(P)$ for all $x \in X$ and $\ker(P)$ is a closed subspace of X. Show that P is continuous.

Problem 4

Let X be a locally compact metric space with Borel σ-algebra $\mathcal{B}_X$. Recall that the topological support of a positive measure μ is defined as

$$\text{supp}(\mu) = \{x \in X \mid \mu(U) > 0 \text{ for all open } U \text{ with } x \in U\}.$$

4a

(10 points) Let $x \in X$ and consider the Dirac measure δ_x on $\mathcal{B}_X$. Show that δ_x is a finite regular measure on $\mathcal{B}_X$ with support equal to $\{x\}$. Determine the functional $f \mapsto \int_X f d\delta_x$, where $f \in C_0(X)$.

4b

(10 points) Suppose that a finite regular measure μ on $\mathcal{B}_X$ is such that $\text{supp}(\mu) = \{x\}$. Prove that $\mu = C\delta_x$ for some real constant $C > 0$.

Problem 5

5a

(10 points) Formulate the Hahn-Banach extension theorem for real vector spaces and for complex vector spaces.

(Continued on page 3.)

5b

(10 points) Let λ be Lebesgue measure on $[0,1]$ and let $L^1([0,1],\lambda)$ be the Banach space of complex-valued integrable functions on $[0,1]$. Suppose that $\{f_k\}_{k\geq 1}$ is a sequence of elements in $L^1([0,1],\lambda)$ and $\{a_k\}_{k\geq 1}$ is a sequence of complex numbers such that there exists $M>0$ with the property that

$$|\sum_{k=1}^{m}\alpha_k a_k| \leq M\big(\int_{[0,1]}|\sum_{k=1}^{m}\alpha_k f_k(x)|d\lambda(x)\big)$$

for any finite sequence of complex numbers $\alpha_1,\ldots,\alpha_m$ and any $m\geq 1$. Show that there is $g\in L^\infty([0,1],\lambda)$ such that

$$\int_{[0,1]} f_k(x)g(x)d\lambda(x) = a_k,\ \forall k\geq 1.$$

END