BONUS EXERCISE

Excercise 1 (Positivity of certain matrices). Let $\mathcal{H}$ denote a complex Euclidean space.

(1) For $x \in \mathbb{C}$ consider the operator $A \in B\left(\mathbb{C}^2\right)$ given by

$$A = \begin{pmatrix} 1 & \bar{x} \\ x & 1 \end{pmatrix}.$$

Show that A is positive if and only if $|x| \leqslant 1$.

(2) For $X \in B(\mathcal{H})$ consider the operator $B \in B\left(\mathcal{H} \oplus \mathcal{H}\right)$ given by the matrix

$$B = \begin{pmatrix} \mathbb{1}_{\mathcal{H}} & X^{\dagger} \\ X & \mathbb{1}_{\mathcal{H}} \end{pmatrix}.$$

Show that B is positive if and only if the operator norm $\|X\| \leqslant 1$.

(3) Show that an operator $C \in B\left(\mathcal{H}\right)$ is positive if and only if the operator

$$\begin{pmatrix} C & C \\ C & C \end{pmatrix} \in B\left(\mathcal{H} \oplus \mathcal{H}\right)$$

is positive.