Quantum information theory (MAT4430) **Spring 2021**

Lecture 13: Additivity problems

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Due to the examples of quantum channels for which the Holevo information is additive, it was conjecture for some time that it might be always additive. This conjecture was disproved by Matt Hastings in 2009, and we will review some ideas behind this result in this and the following lecture.

1 Holevo information and minimum output entropy

Recall the minimum output entropy, which we introduced in the exercises.

Definition 1.1 (Minimal output entropy)**.** *For a quantum channel $T : B(\mathcal{H}_A) \rightarrow B(\mathcal{H}_B)$ we define the* minimum output entropy *as*

$$H_{\min}(T) = \min\left(H(T(\rho)) \ : \ \rho \in D(\mathcal{H}_A)\right).$$

By concavity of the von-Neumann entropy it is easy to see that

$$H_{\min}(T) = \min\left(H(T(|\psi\rangle\langle\psi|)) \ : \ |\psi\rangle\langle\psi| \in \text{Proj}(\mathcal{H}_A)\right).$$

We will now relate the Holevo information to the minimal output entropy. To do so, we will associate a particular quantum channel $\tilde{T}$ to a given quantum channel T.

Definition 1.2. *Let $T : B(\mathcal{H}_A) \rightarrow B(\mathcal{H}_B)$ denote a quantum channel. For $N = \dim(\mathcal{H}_B)^2$, we consider a set of unitaries*

$$\{U_n \ : \ n \in \{1, \ldots, N\}\},$$

such that

$$\frac{\mathbb{1}_{\mathcal{H}_B}}{\dim(\mathcal{H}_B)} \text{Tr} = \frac{1}{N} \sum_{n=1}^{N} \text{Ad}_{U_n}.$$

Then, we define a quantum channel $\tilde{T} : B\left(\mathbb{C}^N \otimes \mathcal{H}_A\right) \rightarrow B(\mathcal{H}_B)$ by

$$\tilde{T}(X) = \sum_{n=1}^{N} \text{Ad}_{U_n} \circ T\left(\left(\langle n| \otimes \mathbb{1}_{\mathcal{H}_A}\right) X \left(|n\rangle \otimes \mathbb{1}_{\mathcal{H}_A}\right)\right).$$

We proved the following theorem in the exercises:

Theorem 1.3. *For any quantum channel $T : B(\mathcal{H}_A) \rightarrow B(\mathcal{H}_B)$ we have*

$$\chi\left(\tilde{T}\right) = \log\left(\dim(\mathcal{H}_B)\right) - H_{\min}(T).$$

The relevance of the previous theorem becomes clear by the following corollary:

Corollary 1.4. *If there exist quantum channels $T : B(\mathcal{H}_A) \rightarrow B(\mathcal{H}_B)$ and $S : B(\mathcal{H}_C) \rightarrow B(\mathcal{H}_D)$ such that*

$$H_{\min}(T \otimes S) < H_{\min}(T) + H_{\min}(S),$$

then we have

$$\chi\left(\tilde{T} \otimes \tilde{S}\right) > \chi\left(\tilde{T}\right) + \chi\left(\tilde{S}\right).$$

Proof. Let $N = \dim(\mathcal{H}_B)^2$ and $M = \dim(\mathcal{H}_D)^2$ and assume that

$$H_{\min}(T \otimes S) < H_{\min}(T) + H_{\min}(S).$$

Consider two sets of unitaries

$$\{U_n \ : \ n \in \{1, \dots, N^2\}\} \text{ and } \{V_m \ : \ m \in \{1, \dots, M^2\}\}$$

such that

$$\frac{\mathbb{1}_{\mathcal{H}_B}}{\dim(\mathcal{H}_B)} \operatorname{Tr} = \frac{1}{N} \sum_{n=1}^{N} \operatorname{Ad}_{U_n}.$$

and

$$\frac{\mathbb{1}_{\mathcal{H}_D}}{\dim(\mathcal{H}_D)} \operatorname{Tr} = \frac{1}{M} \sum_{m=1}^{M} \operatorname{Ad}_{V_m}.$$

Clearly, we have

$$\frac{\mathbb{1}_{\mathcal{H}_B \otimes \mathcal{H}_D}}{\dim(\mathcal{H}_B)\dim(\mathcal{H}_D)} \operatorname{Tr} = \frac{1}{NM} \sum_{n,m} \operatorname{Ad}_{U_n} \otimes \operatorname{Ad}_{V_m},$$

and therefore, we have

$$T \tilde{\otimes} S = \tilde{T} \otimes \tilde{S},$$

for the choice of unitaries from above. Applying Theorem 1.3 (twice) leads to

$$\begin{aligned}
\chi\left(\tilde{T} \otimes \tilde{S}\right) &= \log(\dim(\mathcal{H}_B)) + \log(\dim(\mathcal{H}_D)) - H_{\min}(T \otimes S) \\
&> \log(\dim(\mathcal{H}_B)) - H_{\min}(T) + \log(\dim(\mathcal{H}_D)) - H_{\min}(S) \\
&= \chi(T) + \chi(S).
\end{aligned}$$

This finishes the proof. □

The previous corollary shows that a counterexample to the additivity of the minimum output entropy implies a counterexample to the additivity conjecture of the Holevo quantity. The next corollary shows that it is enough to find two distinct quantum channels for which the minimum output entropy is not additive, since they give rise to a single quantum channel violating the additivity alone.

Corollary 1.5. *If there are quantum channels $S_1, S_2 : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ such that*

$$H_{\min}(S_1 \otimes S_2) < H_{\min}(S_1) + H_{\min}(S_2),$$

then there exists a quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ such that

$$H_{\min}(T \otimes T) < 2H_{\min}(T).$$

Proof. Consider $\rho_1, \rho_2 \in D(\mathcal{H}_A)$ such that

$$H_{\min}(S_1) = H(S_1(\rho_1)), \quad \text{and} \quad H_{\min}(S_2) = H(S_2(\rho_2)).$$

Next, we define quantum channels $T_1, T_2 : B(\mathcal{H}_A) \to B(\mathcal{H}_B)^{\otimes 2}$ by

$$T_1(X) = S_1(X) \otimes S_2(\rho_2), \quad \text{and} \quad T_2(X) = S_1(\rho_1) \otimes S_2(X),$$

and note that

$$H_{\min}(T_1) = H_{\min}(T_2) = H_{\min}(S_1) + H_{\min}(S_2).$$

Moreover, we have

$$\begin{aligned} H_{\min}(T_1 \otimes T_2) &= H_{\min}(S_1 \otimes S_2) + H_{\min}(S_1) + H_{\min}(S_2) \\ &< 2H_{\min}(S_1) + 2H_{\min}(S_2) = \end{aligned}$$

Finally, we consider the direct sum

$$T = T_1 \oplus T_2 : B\left(\mathbb{C}^2 \otimes \mathcal{H}_A\right) \to B\left(\mathbb{C}^2 \otimes \mathcal{H}_B^{\otimes 2}\right)$$

defined by

$$T(X) = T_1\left(\left(\langle 1| \otimes \mathbb{1}_{\mathcal{H}_A}\right) X \left(|1\rangle \otimes \mathbb{1}_{\mathcal{H}_A}\right)\right) \otimes |1\rangle\langle 1| + T_2\left(\left(\langle 2| \otimes \mathbb{1}_{\mathcal{H}_A}\right) X \left(|2\rangle \otimes \mathbb{1}_{\mathcal{H}_A}\right)\right) \otimes |2\rangle\langle 2|.$$

Given a quantum state $\sigma \in D\left(\mathbb{C}^2 \otimes \mathcal{H}_A\right)$ we have

$$T(\sigma) = p_1 T_1(\sigma_1) \otimes |1\rangle\langle 1| + p_2 T_2(\sigma_2) \otimes |2\rangle\langle 2|,$$

where

$$\begin{aligned} p_1 &= \mathrm{Tr}\left[\left(\langle 1| \otimes \mathbb{1}_{\mathcal{H}_A}\right) \sigma \left(|1\rangle \otimes \mathbb{1}_{\mathcal{H}_A}\right)\right] \\ p_2 &= \mathrm{Tr}\left[\left(\langle 2| \otimes \mathbb{1}_{\mathcal{H}_A}\right) \sigma \left(|2\rangle \otimes \mathbb{1}_{\mathcal{H}_A}\right)\right] \end{aligned}$$

and

$$\begin{aligned} \sigma_1 &= \frac{\left(\langle 1| \otimes \mathbb{1}_{\mathcal{H}_A}\right) \sigma \left(|1\rangle \otimes \mathbb{1}_{\mathcal{H}_A}\right)}{p_1} \in D(\mathcal{H}_A) \\ \sigma_2 &= \frac{\left(\langle 2| \otimes \mathbb{1}_{\mathcal{H}_A}\right) \sigma \left(|2\rangle \otimes \mathbb{1}_{\mathcal{H}_A}\right)}{p_2} \in D(\mathcal{H}_A). \end{aligned}$$

We conclude that

$$H\left(T(\sigma)\right) = H(p) + p_1 H(T_1(\sigma_1)) + p_2 H(T_2(\sigma_2)),$$

such that

$$H_{\min}(T) = H_{\min}(T_1) = H_{\min}(T_2).$$

Finally, note that for any $\tau \in D(\mathcal{H})$ (with a particular reordering of tensor factors) we have

$$H\left((T \otimes T)\left(|1\rangle\langle 1| \otimes |2\rangle\langle 2| \otimes \tau\right)\right) = |1\rangle\langle 1| \otimes |2\rangle\langle 2| \otimes (T_1 \otimes T_2)(\tau).$$

Therefore, we have

$$H_{\min}(T \otimes T) \leq H_{\min}(T_1 \otimes T_2) < H_{\min}(T_1) + H_{\min}(T_2) = 2H_{\min}(T).$$

□

2 A family of additivity problems

We will need a few constructions that are best understood when applied in a concrete setting. Here, we will consider a related additivity problem involving so-called Renyi entropies based on Schatten p-norms. For $p > 2$ this problem can be solved without the heavy machinery that is needed for the additivity problem of the minimum output entropy.

2.1 Minimal output Renyi entropies

Before studying the additivity problem for the minimal output entropy, we will look at a slightly easier problem. For any $p > 1$, we may define the *p-Renyi entropy* as

$$H_p(\rho) = \frac{p}{1-p} \log\left(\|\rho\|_p\right),$$

for any quantum state $\rho \in D(\mathcal{H})$. The following lemma shows that these quantities are closely related to the von-Neumann entropy:

Lemma 2.1. *We have*

$$\lim_{p \searrow 1} H_p(\rho) = H(\rho),$$

for any quantum state $\rho \in D(\mathcal{H})$.

Proof. Note that

$$H_p(\rho) = \frac{1}{1-p} \log\left(\|\rho\|_p^p\right),$$

and

$$\|\rho\|_p^p = \sum_{i=1}^{d} \lambda_i^p,$$

where λ_i denote the eigenvalues of ρ. Using l'Hospital's rule we find

$$\lim_{p \searrow 1} H_p(\rho) = \frac{1}{\ln(2)} \lim_{p \searrow 1} -\sum_{i=1}^{d} \ln(\lambda_i)\lambda_i^p = H(\rho).$$

□

Similar to the minimum output entropy, we may define the *minimum output Renyi-entropies* of a quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ as

$$H_p^{(\min)}(T) = \min_{\rho \in D(\mathcal{H}_A)} H_p\left(T(\rho)\right).$$

By monotonicity of the logarithm and since $p > 1$, we can write

$$H_p^{(\min)}(T) = \frac{p}{1-p} \log\left(\max_{\rho \in D(\mathcal{H}_A)} \|T(\rho)\|_p\right),$$

which motivates the definition of the *maximum output p-norm* given by

$$\|T\|_{1\to p} = \max_{\|X\|_1 = 1} \|T(X)\|_p.$$

Note that here we optimize over the full $\|\cdot\|_1$-unit ball, but actually the optimization can be restricted to selfadjoint operators, and then to pure quantum states (i.e., the selfadjoint extreme points of the $\|\cdot\|_1$-unit ball up to a phase factor), showing that

$$H_p^{(\min)}(T) = \frac{p}{1-p} \log\left(\|T\|_{1\to p}\right), \tag{1}$$

for any quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$. We state the following lemma without proof.

Lemma 2.2. *For any quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ we have*

$$\|T\|_{1\to p} = \max_{X \in B(\mathcal{H}_A)_{sa}, \|X\|_1 = 1} \|T(X)\|_p = \max_{|\psi\rangle \in \mathcal{H}_A, \langle\psi|\psi\rangle = 1} \|T(|\psi\rangle\langle\psi|)\|_p.$$

By (1), we see that the following statements are equivalent:

1. (**Additivity of $H_p^{(\min)}$**)
$$H_p^{(\min)}(T \otimes S) = H_p^{(\min)}(T) + H_p^{(\min)}(S).$$

2. (**Multiplicativity of $\|\cdot\|_{1\to p}$**)
$$\|T \otimes S\|_{1\to p} = \|T\|_{1\to p}\|S\|_{1\to p}.$$

Moreover, if $H_p^{(\min)}$ would be additive for all values of p close to 1, then additivity of the minimal output entropy would follow by Lemma 2.1. Historically, this was a strong motivation for studying multiplicativity of the norms $\|\cdot\|_{1\to p}$, and we will review a few key results below.

2.2 Quantum channels and subspaces of tensor products

Consider a subspace $\mathcal{S} \subseteq \mathbb{C}^{d_B} \otimes \mathbb{C}^{d_E}$ and an isometry $V : \mathbb{C}^{d_A} \to \mathbb{C}^{d_B} \otimes \mathbb{C}^{d_E}$ with $d_A = \dim(\mathcal{S})$ and $\mathrm{Im}\,(V) = \mathcal{S}$. We can define a quantum channel $T : B(\mathbb{C}^{d_A}) \to B(\mathbb{C}^{d_B})$ by the Stinespring dilation
$$T(X) = \mathrm{Tr}_E\left[VXV^\dagger\right].$$

In this way, we obtain a correspondence between quantum channels and subspaces of $\mathbb{C}^{d_B} \otimes \mathbb{C}^{d_E}$. We will now use this correspondence to study the additivity problem for maximum output p-norms. We start with a lemma:

Lemma 2.3. *Consider an isometry $V : \mathbb{C}^{d_A} \to \mathbb{C}^{d_B} \otimes \mathbb{C}^{d_E}$ with $Im\,(V) = \mathcal{S} \subseteq \mathbb{C}^{d_B} \otimes \mathbb{C}^{d_E}$. The quantum channel $T : B(\mathbb{C}^{d_A}) \to B(\mathbb{C}^{d_B})$ given by*
$$T(X) = \mathrm{Tr}_E\left[VXV^\dagger\right],$$
satisfies
$$\|T\|_{1\to p} = \max_{|\psi\rangle\in\mathcal{S},\langle\psi|\psi\rangle=1} \|\,\mathrm{Tr}_E\left(|\psi\rangle\langle\psi|\right)\|_p = \max_{\substack{X\in B(\mathbb{C}^{d_E},\mathbb{C}^{d_B}),\|X\|_2=1,\\ \mathrm{vec}(X)\in\mathcal{S}}} \|X\|_{2p}^2.$$

Proof. By Lemma 2.2, we have
$$\|T\|_{1\to p} = \max_{|\phi\rangle\in\mathcal{H}_A,\langle\phi|\phi\rangle=1} \|\,\mathrm{Tr}_E\left[V|\psi\rangle\langle\psi|V^\dagger\right]\|_p.$$

Since $|\psi\rangle = V|\phi\rangle \in \mathcal{S}$ and since any vector $|\psi\rangle \in \mathcal{S}$ can be obtained in this way from some $|\phi\rangle \in \mathcal{H}_A$, we conclude that
$$\|T\|_{1\to p} = \max_{|\psi\rangle\in\mathcal{S},\langle\psi|\psi\rangle=1} \|\,\mathrm{Tr}_E\left(|\psi\rangle\langle\psi|\right)\|_p$$

For the second equality we recall that
$$\mathrm{Tr}_E\left[|\psi\rangle\langle\psi|\right] = XX^\dagger,$$

whenever $\mathrm{vec}(X) = |\psi\rangle$. Since the operator-vector correspondence is an isometric isomorphism and $\|XX^\dagger\|_p = \|X\|_{2p}^2$ for any $X \in B(\mathcal{H}_E, \mathcal{H}_B)$ we conclude that the second equality holds. □

2.3 Conjugate pairs and non-multiplicativity for $p > 2$

Definition 2.4 (Conjugate pairs of quantum channels). *Consider a quantum channel $T : B(\mathbb{C}^{d_A}) \to B(\mathbb{C}^{d_B})$ with Stinespring dilation*

$$T(X) = \text{Tr}_E \left[V X V^\dagger \right],$$

where $V : \mathbb{C}^{d_A} \to \mathbb{C}^{d_B} \otimes \mathbb{C}^{d_E}$ is an isometry. We define the conjugate quantum channel[1] *$\overline{T} : B(\mathbb{C}^{d_A}) \to B(\mathbb{C}^{d_B})$ by*

$$\overline{T}(X) = \text{Tr}_E \left[\overline{V} X \overline{V}^\dagger \right],$$

where $\overline{V}$ denotes the entrywise conjugation of V in the computational basis.

Note that the definition of the conjugate quantum channel depends on the Stinespring dilation, which in general is not unique. The results, which we will prove below will be valid for any conjugate quantum channel and we will usually specify the Stinespring dilation used to define it.

Lemma 2.5 (Spectral property). *Let $T : B(\mathbb{C}^{d_A}) \to B(\mathbb{C}^{d_B})$ denote a quantum channel with Stinespring dilation*

$$T(X) = \text{Tr}_E \left[V X V^\dagger \right],$$

where $V : \mathbb{C}^{d_A} \to \mathbb{C}^{d_B} \otimes \mathbb{C}^{d_E}$ is an isometry. For any $p \geq 1$ we have

$$\| \left(T \otimes \overline{T}\right)(\omega_{A'A}) \|_p \geq \| \left(T \otimes \overline{T}\right)(\omega_{A'A}) \|_\infty \geq \frac{d_A}{d_B d_E},$$

for the conjugate quantum channel $\overline{T} : B(\mathbb{C}^{d_A}) \to B(\mathbb{C}^{d_B})$, where $\omega_{A'A} \in D(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_A})$ denotes the maximally entangled state given by $\omega_{A'A} = |\Omega_{A'A}\rangle\langle\Omega_{A'A}|$ with

$$|\Omega_{A'A}\rangle = \frac{1}{\sqrt{d_A}} \sum_{i=1}^{d_A} |i_A\rangle \otimes |i_A\rangle \in \mathcal{H}_A \otimes \mathcal{H}_A.$$

Proof. Consider the maximally entangled states

$$|\Omega_{B'B}\rangle = \frac{1}{\sqrt{d_B}} \sum_{i=1}^{d_B} |i_B\rangle \otimes |i_B\rangle \in \mathcal{H}_B \otimes \mathcal{H}_B,$$
$$|\Omega_{E'E}\rangle = \frac{1}{\sqrt{d_E}} \sum_{i=1}^{d_E} |i_E\rangle \otimes |i_E\rangle \in \mathcal{H}_E \otimes \mathcal{H}_E.$$

Since $\mathbb{1}_{E'E} \geq |\Omega_{E'E}\rangle\langle\Omega_{E'E}|$, we have (with some ordering of tensor factors)

$$\begin{aligned}
\| \left(T \otimes \overline{T}\right)(\omega_{d_A}) \|_\infty &\geq \langle\Omega_{B'B}| \left(T \otimes \overline{T}\right)(\omega_{A'A}) |\Omega_{B'B}\rangle \\
&= \text{Tr}\left[\left(|\Omega_{B'B}\rangle\langle\Omega_{B'B}| \otimes \mathbb{1}_{E'E}\right) \left(\text{Ad}_V \otimes \text{Ad}_{\overline{V}}\right)(\omega_{A'A}) \right] \\
&\geq \text{Tr}\left[\left(|\Omega_{B'B}\rangle\langle\Omega_{B'B}| \otimes |\Omega_{E'E}\rangle\langle\Omega_{E'E}|\right) \left(\text{Ad}_V \otimes \text{Ad}_{\overline{V}}\right)(\omega_{A'A}) \right] \\
&= \left| \left(\langle\Omega_{B'B}| \otimes \langle\Omega_{E'E}|\right) \left(V \otimes \overline{V}\right) |\Omega_{A'A}\rangle \right|^2 \\
&= \frac{1}{d_E d_A d_B} \left| \text{Tr}\left[V^\dagger V \right] \right|^2 = \frac{1}{d_E d_A d_B} \left| \text{Tr}\left[\mathbb{1}_{\mathbb{C}^{d_A}} \right] \right|^2 = \frac{d_A}{d_E d_B},
\end{aligned}$$

where we used the necklace identity. □

[1] This should not be confused with the complementary channel, which is sometimes referred to as the "conjugate channel" as well.

The previous lemma can be used to show that the quantities $T \mapsto \|T\|_{1\to p}$ are not multiplicative. For this we will need the following lemma, which we will not prove[2]:

Lemma 2.6. *Any $X \in B(\mathbb{C}^d)$ satisfying $X^T = -X$ has even rank, and its non-zero singular values come in pairs, i.e., we have*

$$\sigma_1 = s_1(X) = s_2(X) \geq \sigma_2 = s_3(X) = s_4(X) \geq \cdots \geq \sigma_{r/2} = s_{r-1}(X) = s_r(X) > 0.$$

With this, we can prove:

Theorem 2.7. *For any $p > 2$ there exists a complex Euclidean space $\mathcal{H}$ and quantum channels $T : B(\mathcal{H}) \to B(\mathcal{H})$ and $S : B(\mathcal{H}) \to B(\mathcal{H})$ such that*

$$\|T \otimes S\|_{1\to p} > \|T\|_{1\to p}\|S\|_{1\to p}.$$

Proof. Fix some $p > 2$. Consider the antisymmetric subspace

$$\mathcal{A}_d = \{|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d \ : \ \mathbb{F}|\psi\rangle = -|\psi\rangle\},$$

where $\mathbb{F} \in B(\mathbb{C}^d \otimes \mathbb{C}^d)$ is the flip operator, defined by $\mathbb{F}\left(|i\rangle \otimes |j\rangle\right) = |j\rangle \otimes |i\rangle$ for all i, j. Clearly, we have $\dim(\mathcal{A}_d) = d(d-1)/2$. Using the operator-vector correspondence it is easy to see that

$$\mathcal{A}_d = \{\mathrm{vec}\left(X\right) \ : \ X \in B(\mathbb{C}^d), X^T = -X\}.$$

By Lemma 2.6, we have

$$\|X\|_2^2 = 2\|(\sigma_1, \ldots, \sigma_{r/2})\|_2^2 \geq 2^{1-\frac{2}{p}}\|X\|_p^2.$$

By Lemma 2.3 we conclude that

$$\|T\|_{1\to p} = \max_{\substack{X \in B(\mathbb{C}^d), \|X\|_2 = 1,\\ X^T = -X}} \|X\|_{2p}^2 \leq 2^{\frac{1}{p}-1},$$

for the quantum channel $T : B\left(\mathbb{C}^{d(d-1)/2}\right) \to B\left(\mathbb{C}^d\right)$ associated to $\mathcal{A}_d$ as explained in the beginning of Section 2.2. Using Lemma 2.5 and the ordering of the Schatten p-norms we have

$$\|T \otimes \overline{T}\|_{1\to p} \geq \|T \otimes \overline{T}\|_{1\to\infty} \geq \frac{1}{2}\left(1 - \frac{1}{d}\right) > \frac{1}{2} \cdot 2^{\frac{2}{p}-1} \geq \|T\|_{1\to p}\|\overline{T}\|_{1\to p},$$

whenever the dimension d is large enough. □

[2] For a proof see Corollary 4.4.19. in “Matrix analysis” by Roger Horn and Charles R. Johnston