Quantum information theory (MAT4430) **Spring 2021**

Lecture 17: The quantum capacity and channels for which it vanishes

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In this lecture we will start our discussion of the quantum capacity. Similar to the classical capacity, the quantum capacity quantifies what communication rates are achievable for the transmission of quantum information, i.e., of quantum states preserving their entanglement with any reference system. To measure the communication error it will be convenient to use the diamond-norm, which we introduced in the exercises.

1 Channel fidelity and diamond norm

Recall the so-called *diamond norm*, which we introduced in the exercises:

Definition 1.1 (Diamond norm). *The* diamond norm *of a linear map $L : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ is defined as*

$$\|L\|_\diamond = \sup_{n\in\mathbb{N}} \|\text{id}_n \otimes L\|_{1\to 1},$$

where $\text{id}_n : B(\mathbb{C}^n) \to B(\mathbb{C}^n)$ denotes the identity map.

The following properties where proved in the exercises:

- For any linear map $L : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ we have
$$\|L\|_\diamond = \|\text{id}_{B(\mathcal{H}_A)} \otimes L\|_{1\to 1} \leq \dim(\mathcal{H}_A)\|L\|_{1\to 1}.$$
- For linear maps $L_1 : B(\mathcal{H}_{A_1}) \to B(\mathcal{H}_{B_1})$ and $L_2 : B(\mathcal{H}_{A_2}) \to B(\mathcal{H}_{B_2})$ we have
$$\|L_1 \otimes L_2\|_\diamond = \|L_1\|_\diamond \|L_2\|_\diamond.$$
- The transpose map $\vartheta_d : B(\mathbb{C}^d) \to B(\mathbb{C}^d)$ satisfies
$$\|\vartheta_d\|_\diamond = d.$$

We will often need to relate different measures of the distance between a quantum channel and the identity channel. To prove the necessary estimates we will start with a lemma:

Lemma 1.2. *For any linear map $L : B(\mathcal{H}) \to B(\mathcal{H})$ we have*

$$\|L\|_{1\to 1} \leq 2\max\{\|L(|v\rangle\langle v|)\|_1 \ : \ |v\rangle \in \mathcal{H}, \langle v|v\rangle = 1\}.$$

Recall from Lecture 7 that there is a stronger statement when L is assumed to be positive. Unfortunately, we need to deal with non-positive maps in the following.

Proof. Consider an operator $X \in B(\mathcal{H})$ satisfying $\|X\|_1 = 1$. Then, we have the decomposition

$$X = H_1 + iH_2,$$

with selfadjoint operators $H_1, H_2 \in B(\mathcal{H})_{sa}$ given by

$$H_1 = \frac{1}{2}\left(X + X^\dagger\right),$$
$$H_2 = \frac{1}{2i}\left(X - X^\dagger\right).$$

It is easy to see that $\|H_1\|_1 \leq 1$ and $\|H_2\|_1 \leq 1$, and that

$$\|L(X)\|_1 \leq \|L(H_1)\|_1 + \|L(H_2)\|_1. \tag{1}$$

Next, we consider the spectral decomposition

$$H_1 = \sum_{i=1}^{\dim(\mathcal{H})} \lambda_i |v_i\rangle\langle v_i|,$$

with eigenvalues $\lambda_i \in \mathbb{R}$ and normalized vectors $|v_i\rangle \in \mathcal{H}$. Note that

$$\sum_{i=1}^{\dim(\mathcal{H})} |\lambda_i| = \|H_1\|_1 \leq 1,$$

and by the triangle inequality we conclude that

$$\|L(H_1)\|_1 \leq \sum_{i=1}^{\dim(\mathcal{H})} |\lambda_i| \|L(|v_i\rangle\langle v_i|)\|_1 \leq \max\{\|L(|v\rangle\langle v|)\|_1 \ : \ |v\rangle \in \mathcal{H}, \langle v|v\rangle = 1\}.$$

Repeating the same argument for H_2 and combining these estimates with (1) finishes the proof.

□

Using the previous estimate, we can show the following lemma:

Lemma 1.3. *Let $\mathcal{H}$ denote a complex Euclidean space and $T : B(\mathcal{H}) \to B(\mathcal{H})$ a quantum channel. If we have*

$$F(|v\rangle\langle v|, T(|v\rangle\langle v|)) \geq 1 - \epsilon,$$

for every normalized $|v\rangle \in \mathcal{H}$, then we have

$$\|\mathrm{id}_{B(\mathcal{H})} - T\|_\diamond \leq 8(2\epsilon)^{1/4}.$$

Proof. To prove this statement, we will convert several times between different fidelities and trace norms. By the Fuchs-van-de-Graaf inequalities and Bernoulli's inequality we have

$$1 - \frac{1}{4}\||v\rangle\langle v| - T(|v\rangle\langle v|)\|_1^2 \geq (1-\epsilon)^2 \geq 1 - 2\epsilon,$$

and hence

$$\||v\rangle\langle v| - T(|v\rangle\langle v|)\|_1 \leq 2\sqrt{2\epsilon},$$

for every normalized $|v\rangle \in \mathcal{H}$. By Lemma 1.2 we find that

$$\|\mathrm{id}_{B(\mathcal{H})} - T\|_{1\to 1} \leq 4\sqrt{2\epsilon}.$$

We could now use a result from the exercises to derive an upper bound on the diamond norm. Unfortunately, this bound contains a dimension factor, and to obtain a better bound, we will convert back to a fidelity. Consider any $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}$ with $\langle\psi|\psi\rangle = 1$. By the Schmidt decomposition, we can write

$$|\psi\rangle = \sum_{i=1}^{d} \lambda_i |a_i\rangle \otimes |b_i\rangle,$$

where $d = \dim(\mathcal{H})$ and with Schmidt coefficients $\lambda_i \in \mathbb{R}_0^+$ satisfying $\sum_{i=1}^d \lambda_i^2 = 1$ and orthonormal bases $\{|a_i\rangle\}_{i=1}^d$ and $\{|b_i\rangle\}_{i=1}^d$ of $\mathcal{H}$. We can check that

$$\begin{aligned} 1 - F(|\psi\rangle\langle\psi|, (\mathrm{id}_{B(\mathcal{H})} \otimes T)(|\psi\rangle\langle\psi|))^2 &= \langle\psi|\mathrm{id}_{B(\mathcal{H})} \otimes (\mathrm{id}_{B(\mathcal{H})} - T)(|\psi\rangle\langle\psi|)|\psi\rangle \\ &= \sum_{i,j=1}^d \lambda_i^2\lambda_j^2 |\langle b_i| \left(\mathrm{id}_{B(\mathcal{H})} - T\right) (|b_i\rangle\langle b_j|)\, |b_j\rangle| \\ &\leq \|\mathrm{id}_{B(\mathcal{H})} - T\|_{1\to 1} \leq 4\sqrt{2\epsilon}, \end{aligned}$$

where we used that the squares of Schmidt coefficients sum up to 1 and that

$$\begin{aligned} |\langle b_i| \left(\mathrm{id}_{B(\mathcal{H})} - T\right) (|b_i\rangle\langle b_j|)\, |b_j\rangle| &\leq \| \left(\mathrm{id}_{B(\mathcal{H})} - T\right) (|b_i\rangle\langle b_j|) \|_\infty \\ &\leq \| \left(\mathrm{id}_{B(\mathcal{H})} - T\right) (|b_i\rangle\langle b_j|) \|_1 \leq \|\mathrm{id}_{B(\mathcal{H})} - T\|_{1\to 1}, \end{aligned}$$

for any $i, j \in \{1, \ldots, d\}$. Next, we use the Fuchs-van-de-Graaf inequalities again to see that

$$\| \left(\mathrm{id}_{B(\mathcal{H})} \otimes \left(\mathrm{id}_{B(\mathcal{H})} - T\right)\right) (|\psi\rangle\langle\psi|) \|_1 \leq 4(2\epsilon)^{1/4},$$

for every normalized vector $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}$. Finally, we apply Lemma 1.2 and a result from the exercises to conclude

$$\|\mathrm{id}_{B(\mathcal{H})} - T\|_\diamond \leq 8(2\epsilon)^{1/4}.$$

□

It should be noted that the constants in the previous lemma are not optimal. Using some tricks (see Watrous) the final bound can be slightly improved to

$$\|\mathrm{id}_{B(\mathcal{H})} - T\|_\diamond \leq 2(2\epsilon)^{1/4}.$$

However, these constants will not play any role in the following, since we will always consider situations where $\epsilon \to 0$ exponentially fast and where we are interested in the rates of this exponential convergence.

2 Definition of the quantum capacity

As always, we begin with the definition of coding schemes:

Definition 2.1. *Let $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ denote a quantum channel. An (n, m, δ)-coding scheme for* quantum communication *over T is a pair of quantum channels*

$$E : B\left((\mathbb{C}^2)^{\otimes m}\right) \to B(\mathcal{H}_A^{\otimes n}) \quad \textit{and} \quad D : B(\mathcal{H}_B^{\otimes n}) \to B\left((\mathbb{C}^2)^{\otimes m}\right),$$

such that

$$\|\mathrm{id}_2^{\otimes m} - D \circ T^{\otimes n} \circ E\|_\diamond \leq \delta.$$

In Figure 1 you can see a schematic visualizing a coding scheme for the transmission of quantum information.

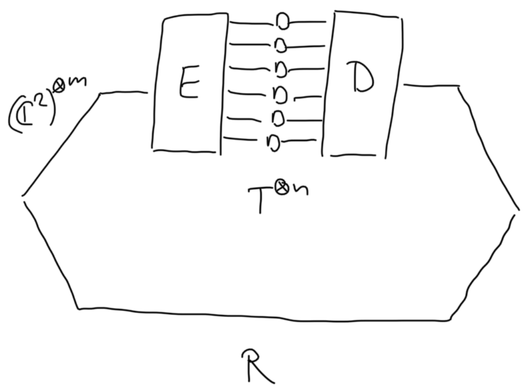

Figure 1: Coding scheme for the transmission of quantum information with reference system

Then, we may define the quantum capacity as follows:

Definition 2.2 (Quantum capacity)**.** *We call a rate $R \geq 0$ achievable for* quantum communication *over the quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ if for every $n \in \mathbb{N}$, there exists an (n, m_n, δ_n)-coding scheme such that*

$$R = \lim_{n\to\infty} \frac{m_n}{n} \quad \text{and} \quad \lim_{n\to\infty} \delta_n = 0.$$

The quantum capacity *of a quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ is given by*

$$Q(T) = \sup\{R \geq 0 \ : \ R \text{ achievable rate for quantum communication}\}.$$

We will omit the proof of the following lemma since it is very similar to a lemma involving the classical capacity of a quantum channel, which was proved in the exercises.

Lemma 2.3. *For any quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ and any $k \in \mathbb{N}$ we have*

$$Q(T^{\otimes k}) = kQ(T).$$

3 Channels with vanishing quantum capacity

We have seen that the constant quantum channels are the only quantum channels with vanishing classical capacity. In the case of the quantum capacity, the situation is very different: There are at least two different reasons for a quantum channel to have zero quantum capacity leading to two distinct classes of quantum channels with vanishing quantum capacity. At the point of writing, it is not known whether there are quantum channels outside of these classes for which the quantum capacity vanishes. Deciding whether there are such examples is a major open problem in quantum Shannon theory!

3.1 The transposition bound

The following theorem gives an upper bound on the quantum capacity of a quantum channel:

Theorem 3.1 (The transposition bound)**.** *For any quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ we have*

$$Q(T) \leq \log\left(\|\vartheta_B \circ T\|_\diamond\right).$$

Proof. Note that the diamond norm of the transpose map $\vartheta_B : B(\mathcal{H}_B) \to B(\mathcal{H}_B)$ is given by $\|\vartheta_B\|_\diamond = \dim(\mathcal{H}_B)$. Consider now an (n, m, δ)-coding scheme for quantum communication over the quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ given by a pair of quantum channels $E : B\left((\mathbb{C}^2)^{\otimes m}\right) \to B\left(\mathcal{H}_A^{\otimes m}\right)$ and $D : B\left(\mathcal{H}_A^{\otimes m}\right) \to B\left((\mathbb{C}^2)^{\otimes m}\right)$ satisfying

$$\delta = \|\mathrm{id}_2^{\otimes m} - D \circ T^{\otimes n} \circ E\|_\diamond.$$

Using multiplicativity of the diamond norm and the triangle inequality, we may compute

$$\begin{aligned} 2^m = \|\vartheta_2\|_\diamond^m = \|\vartheta_2^{\otimes m}\|_\diamond \\ &= \|\vartheta_2^{\otimes m} \circ \left(\mathrm{id}_2^{\otimes m} - D \circ T^{\otimes n} \circ E + D \circ T^{\otimes n} \circ E\right)\|_\diamond \\ &\leq \|\vartheta_2^{\otimes m} \circ \left(\mathrm{id}_2^{\otimes m} - D \circ T^{\otimes n} \circ E\right)\|_\diamond + \|\vartheta_2^{\otimes m} \circ D \circ T^{\otimes n} \circ E\|_\diamond \\ &\leq \delta\|\vartheta_2^{\otimes m}\|_\diamond + \|(\vartheta_2^{\otimes m} \circ D \circ \vartheta_B^{\otimes n}) \circ \vartheta_B^{\otimes n} \circ T^{\otimes n} \circ E\|_\diamond \\ &\leq \delta 2^m + \|\vartheta_B \circ T\|_\diamond^n. \end{aligned}$$

Rearranging this inequality and taking the logarithm shows that

$$\frac{m}{n} + \frac{\log(1-\delta)}{n} \leq \log\left(\|\vartheta_B \circ T\|_\diamond\right).$$

Applying this inequality for a sequence of (n, m_n, δ_n)-coding schemes with

$$\lim_{n\to\infty} \frac{m_n}{n} = R \quad \text{and} \quad \lim_{n\to\infty} \delta_n = 0,$$

gives the inequality

$$R = \lim_{n\to\infty} \left(\frac{m_n}{n} + \frac{\log(1-\delta)}{n}\right) \leq \log\left(\|\vartheta_B \circ T\|_\diamond\right),$$

which finishes the proof. □

Besides giving an upper bound on a complicated quantity, the previous theorem has an important consequence:

Corollary 3.2. *If the composition $\vartheta_B \circ T$ is completely positive for a quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$, then we have $Q(T) = 0$.*

Proof. If $\vartheta_B \circ T$ is completely positive, then we can compute

$$\|\vartheta_B \circ T\|_\diamond = \|\mathrm{id}_{\mathcal{H}_A} \otimes \vartheta_B \circ T\|_{1\to 1} = \|\mathbb{1}_{\mathcal{H}_A} \otimes (T^* \circ \vartheta_B^*)(\mathbb{1}_{\mathcal{H}_B})\|_\infty = \|\mathbb{1}_{\mathcal{H}_A} \otimes \mathbb{1}_{\mathcal{H}_A}\|_\infty = 1,$$

where we used the Russo-Dye theorem and the fact that T is trace-preserving. By Theorem 3.1 we have

$$Q(T) \leq \log\left(\|\vartheta_B \circ T\|_\diamond\right) = 0.$$

□

In Lecture 6 we have seen an example of an entangled quantum state with positive partial transpose. Whenever the Choi operator of a quantum channel yields such a state, then the channel has vanishing quantum capacity. There are many examples of such quantum channels showing that the structure of quantum channels with vanishing quantum capacity is much richer than for the classical capacity. Surprisingly, the channels staying completely positive under composition with the transpose are not even the only quantum channels with zero quantum capacity as we will see now.

3.2 Antidegradable channels have zero quantum capacity

Consider a quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ given in its Stinespring form

$$T(X) = \text{Tr}_E \left(VXV^\dagger \right),$$

for any $X \in B(\mathcal{H}_A)$ and with an isometry $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$. Recall the *complementary channel* $T^c : B(\mathcal{H}_A) \to B(\mathcal{H}_E)$ given by

$$T^c(X) = \text{Tr}_B \left(VXV^\dagger \right),$$

for any $X \in B(\mathcal{H}_A)$. The channel T is called

- *degradable* if there is a quantum channel $S : B(\mathcal{H}_B) \to B(\mathcal{H}_E)$ such that $S \circ T = T^c$.
- *antidegrable* if there is a quantum channel $R : B(\mathcal{H}_E) \to B(\mathcal{H}_B)$ such that $R \circ T^c = T$.

Degradable and antidegradable quantum channels will play an important role in the study of the quantum capacity. For now, we just note the following theorem:

Theorem 3.3. *If the quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ is antidegradable, then we have $Q(T) = 0$.*

Proof. Assume for contradiction that there are quantum channels $E_n : B(\mathbb{C}^2) \to B(\mathcal{H}_A^{\otimes n})$ and $D_n : B(\mathcal{H}_B^{\otimes n}) \to B(\mathbb{C}^2)$ for each $n \in \mathbb{N}$ such that

$$\text{id}_2 = \lim_{n\to\infty} D_n \circ T^{\otimes n} \circ E_n.$$

Such channels can easily be obtained from coding schemes achieving some non-zero rate. Consider now the quantum channels $S_n : B(\mathbb{C}^2) \to B(\mathbb{C}^2 \otimes \mathbb{C}^2)$ given by

$$S_n(X) = (D_n \otimes D_n \circ R) \circ \text{Ad}_V^{\otimes n} \circ E_n,$$

where $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ denotes the Stinespring isometry of T and $R : B(\mathcal{H}_E) \to B(\mathcal{H}_B)$ such that $R \circ T^c = T$. By compactness of the set of quantum channel, there exists a convergent subsequence S_{n_k} and a quantum channel $S = \lim_{k\to\infty} S_{n_k}$. It is now easy to see that $S : B(\mathbb{C}^2) \to B(\mathbb{C}^2 \otimes \mathbb{C}^2)$ is a quantum channel satisfying

$$(\text{id}_2 \otimes \text{Tr}) \circ S = (\text{Tr} \otimes \text{id}_2) \circ S = \text{id}_2,$$

contradicting the no-cloning theorem. □

4 Quantum communication and entanglement generation

To prove the coding theorem for the quantum capacity it will be useful to consider the task of entanglement generation. Instead of transmitting part of an arbitrary quantum state, entanglement generation concerns the creating of a maximally entangled quantum state between two distant parties. At first sight, this seems to be a strictly weaker task than communication, but it will turn out that the corresponding capacities, i.e., the suprema of achievable rates for these two tasks are the same.

Definition 4.1 (Coding schemes for generating entanglement)**.** *Let $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ denote a quantum channel. An (n, m, δ)-coding scheme for* entanglement generation using *T is a quantum channel*

$$D : B(\mathcal{H}_B^{\otimes n}) \to B\left((\mathbb{C}^2)^{\otimes m} \right),$$

together with a quantum state

$$\rho \in D\left((\mathbb{C}^2)^{\otimes m} \otimes (\mathcal{H}_A)^{\otimes n}\right),$$

such that

$$F\left(\omega_2^{\otimes m}, \left(\mathrm{id}_2^{\otimes m} \otimes D \circ T^{\otimes n}\right)(\rho)\right) \geq 1 - \delta.$$

When considering coding schemes for generating entanglement we will often restrict to the case of pure quantum states $\rho = |\psi\rangle\langle\psi|$. We do not loose generality by doing so. Indeed, the function

$$\rho \mapsto F\left(\omega_2^{\otimes m}, \left(\mathrm{id}_2^{\otimes m} \otimes D \circ T^{\otimes n}\right)(\rho)\right)^2,$$

for fixed quantum channels D and T is convex and continuous, and hence its maximum over the set of quantum states is attained in a pure state. Therefore, we do not make a given coding scheme for entanglement generation worse by replacing its quantum state ρ by a pure state.

Definition 4.2 (Entanglement-generation capacity)**.** *We call a rate $R \geq 0$ achievable for* generating entanglement *using the quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ if for every $n \in \mathbb{N}$, there exists an (n, m_n, δ_n)-coding scheme for entanglement generation using T such that*

$$R = \lim_{n\to\infty} \frac{m_n}{n} \quad \textit{and} \quad \lim_{n\to\infty} \delta_n = 0.$$

The entanglement-generation capacity *of a quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ is*

$$Q_{EG}(T) = \sup\{R \geq 0 \ : \ R \textit{ achievable rate for entanglement generation using } T\}.$$

We will now show that this quantity coincides with the quantum capacity. For this, we will need a lemma:

Lemma 4.3. *Let $\mathcal{H}, \mathcal{H}'$ denote complex Euclidean spaces with $\dim(\mathcal{H}), \dim(\mathcal{H}') \geq 2$. Consider a quantum channel $T : B(\mathcal{H}') \to B(\mathcal{H})$ satisfying*

$$F\left(\omega_{\mathcal{H}}, \left(\mathrm{id}_{B(\mathcal{H})} \otimes T\right)(|\phi\rangle\langle\phi|)\right) \geq 1 - \delta,$$

for some pure state $|\phi\rangle \in \mathcal{H} \otimes \mathcal{H}'$ and where $\omega_{\mathcal{H}} \in \mathcal{H} \otimes \mathcal{H}$ denotes the normalized maximally entangled state. Then, for any complex Euclidean space $\mathcal{K}$ with $\dim(\mathcal{K}) \leq \dim(\mathcal{H})/2$, there exist quantum channels

$$E : B(\mathcal{K}) \to B(\mathcal{H}') \quad \textit{and} \quad D : B(\mathcal{H}) \to B(\mathcal{K})$$

such that

$$\|\mathrm{id}_{B(\mathcal{K})} - D \circ T \circ E\|_{\diamond} \leq 8\,(32\delta)^{1/4}.$$

The proof of this lemma will use the following lemma. We will postpone its proof to the exercises:

Lemma 4.4. *Let $\mathcal{H}, \mathcal{H}'$ denote complex Euclidean spaces and consider a quantum channel $T : B(\mathcal{H}') \to B(\mathcal{H})$ satisfying*

$$F\left(\omega_{\mathcal{H}}, \left(\mathrm{id}_{B(\mathcal{H})} \otimes T\right)(|\phi\rangle\langle\phi|)\right) \geq 1 - \delta,$$

for some pure state $|\phi\rangle \in \mathcal{H} \otimes \mathcal{H}'$. Then, there exists a number $r \leq \dim(\mathcal{H})$ and a pure state $|\Omega_r\rangle \in \mathcal{H} \otimes \mathcal{H}$ of the form

$$|\Omega_r\rangle = \frac{1}{\sqrt{r}} \sum_{i=1}^{r} \overline{|a_i\rangle} \otimes |a_i\rangle$$

for some orthonormal set $\{|a_i\rangle\}_{i=1}^r \subset \mathcal{H}$, and a quantum channel $E : B(\mathcal{H}) \to B(\mathcal{H}')$ with

$$F\left(\omega_{\mathcal{H}}, \left(\mathrm{id}_{B(\mathcal{H})} \otimes T \circ E\right)(|\Omega_r\rangle\langle\Omega_r|)\right) \geq 1 - 4\delta.$$

Proof. Exercises. □

Now we can proceed with the proof of the main lemma:

Proof of Lemma 4.3. Using Lemma 4.4 and considering the quantum channel $T \circ E$ (see the statement of Lemma 4.4) instead of T we can restrict to the case where $\mathcal{H}' = \mathcal{H}$ and

$$|\phi\rangle = \frac{1}{\sqrt{r}} \sum_{i=1}^{r} \overline{|a_i\rangle} \otimes |a_i\rangle \in \mathcal{H} \otimes \mathcal{H}$$

for some orthonormal set $\{|a_1\rangle, \dots, |a_r\rangle\} \subset \mathcal{H}$ where we write $d = \dim(\mathcal{H})$. Remember, that by doing so we have

$$F\left(\omega_{\mathcal{H}}, (\mathrm{id}_{B(\mathcal{H})} \otimes T)(|\phi\rangle\langle\phi|)\right) \geq 1 - 4\delta.$$

We start by defining an orthonormal set of vectors $\{|w_1\rangle, \dots, |w_r\rangle\} \subset \mathcal{H}$ generating nested subspaces $\mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \subset \mathcal{V}_r = \mathrm{span}\{|a_1\rangle, \dots, |a_r\rangle\}$ as

$$\mathrm{span}\{|w_1\rangle, \dots, |w_k\rangle\} = \mathcal{V}_k, \text{ for each } k \in \{1, \dots, r\}. \tag{2}$$

Initially, we set $\mathcal{V}_r = \mathrm{span}\{|a_1\rangle, \dots, |a_r\rangle\}$ and we recursively define

$$\begin{cases} \alpha_k = \min\{\langle |v\rangle\langle v|, T(|v\rangle\langle v|)\rangle_{HS} \ : \ |v\rangle \in \mathcal{V}_k, \langle v|v\rangle = 1\} \in \mathbb{R}_0^+ \\ |w_k\rangle \in \mathcal{V}_k \text{ some normalized vector satisfying } \langle |w_k\rangle\langle w_k|, T(|w_k\rangle\langle w_k|)\rangle_{HS} = \alpha_k \\ \mathcal{V}_{k-1} = \{|v\rangle \in \mathcal{V}_k \ : \ \langle w_k|v\rangle = 0\} \subset \mathcal{V}_k, \end{cases}$$

for all k from r to 1. Clearly, the set of vectors $\{|w_1\rangle, \dots, |w_r\rangle\} \subset \mathcal{H}$ constructed in this way satisfies (2). Moreover, since this set forms an orthonormal basis of $\mathrm{span}\{|a_1\rangle, \dots, |a_r\rangle\}$ we have

$$|\phi\rangle\langle\phi| = \frac{1}{r} \sum_{k,l=1}^{r} \overline{|w_k\rangle\langle w_l|} \otimes |w_k\rangle\langle w_l|,$$

since $|\phi\rangle = \mathrm{vec}(\Pi)/\sqrt{r}$, where

$$\Pi = \sum_{i=1}^{r} |a_i\rangle\langle a_i| = \sum_{i=1}^{r} |w_i\rangle\langle w_i|,$$

is the projection onto $\mathrm{span}\{|a_1\rangle, \dots, |a_r\rangle\}$. After extending $\{|w_1\rangle, \dots, |w_r\rangle\}$ to an orthonormal basis $\{|w_1\rangle, \dots, |w_d\rangle\}$ of $\mathcal{H}$, we also have

$$\omega_{\mathcal{H}} = \frac{1}{d} \sum_{k,l=1}^{d} \overline{|w_k\rangle\langle w_l|} \otimes |w_k\rangle\langle w_l|.$$

Using these expressions, we find that

$$F\left(\omega_{\mathcal{H}}, (\mathrm{id}_{B(\mathcal{H})} \otimes T)(|\phi\rangle\langle\phi|)\right)^2 = \frac{1}{dr} \sum_{k,l=1}^{r} \langle |w_k\rangle\langle w_l|, T(|w_k\rangle\langle w_l|)\rangle_{HS}.$$

Using the Kraus decomposition $T = \sum_{n=1}^{N} \mathrm{Ad}_{K_n}$ we find that

$$\begin{aligned} |\langle |w_k\rangle\langle w_l|, T(|w_k\rangle\langle w_l|)\rangle_{HS}| &= |\sum_{n=1}^{N} \langle w_k|K_n|w_k\rangle\langle w_l|K_n|w_l\rangle| \\ &\leq \sqrt{\sum_{n=1}^{N} |\langle w_k|K_n|w_k\rangle|^2} \sqrt{\sum_{n=1}^{N} |\langle w_l|K_n|w_l\rangle|^2} \\ &= \sqrt{\langle |w_k\rangle\langle w_k|, T(|w_k\rangle\langle w_k|)\rangle_{HS}} \sqrt{\langle |w_l\rangle\langle w_l|, T(|w_l\rangle\langle w_l|)\rangle_{HS}} \\ &= \sqrt{\alpha_k \alpha_l}, \end{aligned}$$

and hence that

$$F\left(\omega_{\mathcal{H}}, (\mathrm{id}_{B(\mathcal{H})} \otimes T)\left(|\phi\rangle\langle\phi|\right)\right) \leq \frac{1}{\sqrt{dr}} \sum_{k=1}^{r} \sqrt{\alpha_k}.$$

Using the Cauchy-Schwarz inequality, we conclude that

$$1 - 4\delta \leq F\left(\omega_{\mathcal{H}}, (\mathrm{id}_{B(\mathcal{H})} \otimes T)\left(|\phi\rangle\langle\phi|\right)\right) \leq \frac{1}{\sqrt{dr}} \sum_{k=1}^{r} \sqrt{\alpha_k} \leq \sqrt{\frac{1}{d} \sum_{k=1}^{r} \alpha_k},$$

by which we find that

$$\frac{1}{d} \sum_{k=1}^{r} \alpha_k \geq (1 - 4\delta)^2 \geq 1 - 8\delta.$$

Define the number

$$m = \max\{k \in \{1, \ldots, r\} \ : \ \alpha_k \geq 1 - 16\delta\},$$

and, since $1 \geq \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_r$, we have

$$1 - 8\delta \leq \frac{1}{d} \sum_{k=1}^{m} \alpha_k + \frac{1}{d} \sum_{k=m+1}^{r} \alpha_k \leq \frac{m}{d} + \frac{r-m}{d}(1 - 16\delta) \leq \frac{m}{d} + \frac{d-m}{d}(1 - 16\delta),$$

from which we conclude that $m \geq d/2$. Consider now some complex Euclidean space $\mathcal{K}$ of dimension $\dim(\mathcal{K}) \leq d/2$. There exists an isometry $V : \mathcal{K} \to \mathcal{V}_m \subseteq \mathcal{H}$ and we may define a quantum channel $E : B(\mathcal{K}) \to B(\mathcal{H})$ by $E = \mathrm{Ad}_V$. Furthermore, we may define another quantum channel $D : B(\mathcal{H}) \to B(\mathcal{K})$ by

$$D(X) = V^\dagger X V + \mathrm{Tr}\left[(\mathbb{1}_{\mathcal{H}} - VV^\dagger)X\right] \sigma,$$

for some quantum state $\sigma \in D(\mathcal{K})$. For any $|x\rangle \in \mathcal{K}$ we have

$$F(E(|x\rangle\langle x|), (T \circ E)(|x\rangle\langle x|)) = F(|v\rangle\langle v|, T(|v\rangle\langle v|)) \geq \alpha_m \geq 1 - 16\delta,$$

where $|v\rangle = V|x\rangle \in \mathcal{V}_m$. Using the data-processing inequality of the fidelity and the fact that $(D \circ E)(|x\rangle\langle x|) = |x\rangle\langle x|$ for any $|x\rangle \in \mathcal{K}$, we conclude that

$$\begin{aligned} F(|x\rangle\langle x|, (D \circ T \circ E)(|x\rangle\langle x|)) &= F((D \circ E)(|x\rangle\langle x|), (D \circ T \circ E)(|x\rangle\langle x|)) \\ &\geq F(E(|x\rangle\langle x|), (T \circ E)(|x\rangle\langle x|)) \\ &\geq 1 - 16\delta, \end{aligned}$$

for any $|x\rangle \in \mathcal{K}$. Using Lemma 1.3 we have

$$\|\mathrm{id}_{B(\mathcal{K})} - D \circ T \circ E\|_\diamond \leq 8\,(32\delta)^{1/4}.$$

□

Finally, we can prove that the quantum capacity of a quantum channel coincides with the entanglement-sharing capacity:

Theorem 4.5. *For any quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ we have*

$$Q(T) = Q_{EG}(T).$$

Proof. Fix some quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ and let $R \geq 0$ be an achievable rate for transmitting quantum information over T such that for every $n \in \mathbb{N}$, there exists an (n, m_n, δ_n)-coding scheme with

$$R = \lim_{n\to\infty} \frac{m_n}{n} \quad \text{and} \quad \lim_{n\to\infty} \delta_n = 0.$$

Let us denote by

$$E_n : B\left((\mathbb{C}^2)^{\otimes m_n}\right) \to B(\mathcal{H}_A^{\otimes n}) \quad \text{and} \quad D_n : B(\mathcal{H}_B^{\otimes n}) \to B\left((\mathbb{C}^2)^{\otimes m_n}\right),$$

the coding channels making up the coding schemes such that

$$\|\text{id}_2^{\otimes m_n} - D_n \circ T^{\otimes n} \circ E_n\|_\diamond = \delta_n \to 0,$$

as $n \to \infty$. Now, note that

$$\|\omega_2^{\otimes m_n} - \left(\text{id}_2^{\otimes m_n} \otimes D_n \circ T^{\otimes n} \circ E_n\right)\left(\omega_2^{\otimes m_n}\right)\|_1 \leq \|\text{id}_2^{\otimes m_n} - D_n \circ T^{\otimes n} \circ E_n\|_\diamond \to 0$$

as $n \to \infty$. Using the Fuchs-van-de-Graaf inequalities, we conclude that

$$F\left(\omega_2^{\otimes m_n}, \left(\text{id}_2^{\otimes m_n} \otimes D_n \circ T^{\otimes n} \circ E_n\right)\left(\omega_2^{\otimes m_n}\right)\right) \to 1,$$

as $n \to \infty$. This shows that R is an achievable rate for generating entanglement by using the coding schemes defined by the quantum channels D_n and the quantum states

$$\rho_n = \left(\text{id}_2^{\otimes m_n} \otimes E_n\right)\left(\omega_2^{\otimes m_n}\right).$$

We have shown that

$$Q(T) \leq Q_{EG}(T).$$

To show the other direction, consider (n, m_n, δ_n)-coding schemes for every $n \in \mathbb{N}$ for entanglement sharing over T such that $\delta_n \to 0$ as $n \to \infty$ and achieving a rate

$$R = \lim_{n\to\infty} \frac{m_n}{n}.$$

Assume that the quantum channels

$$D_n : B(\mathcal{H}_B^{\otimes n}) \to B\left((\mathbb{C}^2)^{\otimes m_n}\right),$$

together with pure quantum states

$$|\phi_n\rangle \in (\mathbb{C}^2)^{\otimes m_n} \otimes (\mathcal{H}_A)^{\otimes n}$$

make up these coding schemes such that

$$F\left(\omega_2^{\otimes m_n}, \left(\text{id}_2^{\otimes m_n} \otimes D_n \circ T^{\otimes n}\right)\left(|\phi_n\rangle\langle\phi_n|\right)\right) \geq 1 - \delta_n \to 1,$$

as $n \to \infty$. For every $n \in \mathbb{N}$ we now use Lemma 4.3 with the complex Euclidean space $\mathcal{K} = (\mathbb{C}^2)^{\otimes(m_n-1)}$ and $\mathcal{H} = (\mathbb{C}^2)^{\otimes m_n}$ satisfying $\dim(\mathcal{K}) = \dim(\mathcal{H})/2$. This shows the existence of quantum channels

$$\tilde{E}_n : B((\mathbb{C}^2)^{\otimes(m_n-1)}) \to B(\mathcal{H}_A^{\otimes n}) \quad \text{and} \quad \tilde{D}_n : B((\mathbb{C}^2)^{\otimes m_n}) \to B((\mathbb{C}^2)^{\otimes(m_n-1)})$$

such that

$$\|\text{id}_2^{\otimes(m_n-1)} - \tilde{D}_n \circ D_n \circ T^{\otimes n} \circ \tilde{E}_n\|_\diamond \leq 8\,(32\delta_n)^{1/4} \to 0,$$

as $n \to \infty$. We conclude that the rate

$$R = \lim_{n\to\infty} \frac{m_n - 1}{n},$$

is achievable for quantum communication via the quantum channel T. □